AG+ Fall School

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Let $\mathbb{S}^n_+$ denote the set of $n \times n$ real symmetric positive semidefinite (PSD) matrices in the vector space $\mathbb{S}^n$ of $n \times n$ real symmetric matrices. The vector space $\mathbb{S}^n$ is equipped with the inner product $\langle A, B \rangle = \operatorname{trace} AB$.

Structure of $\mathbb{S}^n_+$:

1. Show the following:
   (1) $\mathbb{S}^n_+$ is a convex cone in $\mathbb{S}^n$.
   (2) The *extreme rays* of $\mathbb{S}^n_+$ are precisely the rank 1 PSD matrices.
   (3) Let L be a linear subspace of $\mathbb{R}^n$ and denote by $\mathcal{F}_L$ the subcone of $\mathbb{S}^n_+$ consisting of matrices whose kernel contains L. Then any *face* F of $\mathbb{S}^n_+$ is equal to $\mathcal{F}_L$ for some L.

Conic Duality:

To a convex cone $C \subset \mathbb{S}^n$ we associate the dual cone C^* defined by:

$$C^* = \{A \in \mathbb{S}^n \ \mid \ \langle A, B \rangle \geq 0 \quad \text{for all} \quad B \in C\}.$$

Important Fact: if C is a closed cone then $(C^*)^* = C$. This is called the bi-duality theorem.

2. Show the following:
   (1) $\mathbb{S}^n_+$ is *self-dual*, i.e. $(\mathbb{S}^n_+)^* = \mathbb{S}^n_+$.

   (2) For a simple graph G on n vertices, let $L(G)$ denote the subspace of $\mathbb{S}^n$ consisting of matrices where entries corresponding to non-edges of G are zero. Let $\Sigma(G)$ be the orthogonal projection of $\mathbb{S}^n_+$ onto $L(G)$. Show that
   $$\Sigma(G)^* = L(G) \cap \mathbb{S}^n_+.$$
   (3) Duality reverses inclusion: If $A \subseteq B$ then $B^* \subseteq A^*$.

   (4) Show that the dual cone $P(G)^*$ of $P(G)$ is equal to the conical hull of rank 1 PSD matrices in $L(G)$. Compare this dual cone with the cone $\Sigma(G)^*$ from above. Formulate what the equality of $P(G)$ and $\Sigma(G)$ implies for the dual cones $P(G)^*$ and $\Sigma(G)^*$.

Chordal graph theorem.

3. Let I be a square-free quadratic monomial ideal in $\mathbb{R}[x_1, \ldots, x_n]$, and let G be the graph on n vertices where monomials of I are the *non-edges* of G. Show that the variety defined by I is a union of coordinate linear spaces corresponding to maximal cliques of G.

Let C_n denote the cycle on n vertices.

4. Show that $\Sigma(C_n) \subsetneq P(C_n)$. Conclude that if G is not chordal then $\Sigma(G) \subsetneq P(G)$. Hint: one possible approach is to consider the dual statement and use Exercise 2 above.

5. Show the following alternative characterization of chordal graphs: G is chordal if and only if it is a *clique sum* of complete graphs. It suffices to show that if G is chordal, then there a vertex of G whose neighbors form a complete graph.

6. Show that if $\Sigma(G_1) = P(G_1)$ and $\Sigma(G_2) = P(G_2)$ then $\Sigma(G) = P(G)$, where G is a clique sum of G_1 and G_2.

Nonnegative Polynomials and Sums of Squares:

Let $\mathbb{R}[x]_d$ be the vector space of real homogeneous polynomials (forms) in n variables of degree d.

7. For a polynomial $p(x_1, \dots, x_n)$ in n variables of degree d, define *homogenization* $\bar{p}$ of p with respect to a new variable x_0 to be

$$\bar{p}(x_0, \dots, x_n) = x_0^d\, p\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

For example, homogenization of $x_1 + x_2^2$ is $x_0x_1 + x_2^2$ and homogenization of $x_1^3 - x_1x_2 + 3$ is $x_1^3 - x_0x_1x_2 + 3x_0^3$. Show that if a polynomial p is nonnegative then its homogenization is nonnegative as well. Does the same hold for strictly positive polynomials?

8. The *Newton Polytope* N_p of a polynomial p is the convex hull of the vectors of monomial exponents that occur in p. For example, the Newton Polytope of $x^2 + xy + z^2$ is the convex hull of vectors $(2,0,0), (1,1,0)$ and $(0,0,2)$.

Show that if a polynomial $p = \sum_i q_i^2$ is a sum of squares then the Newton Polytope of each q_i is contained in $\frac{1}{2}N_p$.

9. Use Exercise 8 to show that the Motzkin polynomial $M = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$ is not a sum of squares.

10. (Another perspective on Motzkin's construction). Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^3$ be the map sending $[x : y : z]$ to $[x^2y : xy^2 : z^3 : xyz]$. Let $X \subset \mathbb{P}^3$ be (Zariski closure of) the image of φ.

(1) Show that X is a cubic hypersurface in $\mathbb{P}^3$. Conclude that $\Sigma_X \subsetneq P_X$.
(2) Use Exercise 8 and part (1) to show that there exist ternary sextics that are not sums of squares.

11. (Choi-Lam) Show that the form $S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$ is nonnegative but not a sum of squares.

12. Show that the maximal faces of $P_{n,2d}$ (with respect to inclusion) have the form

$$F_v = \{f \in P_{n,2d} \mid f(v) = 0 \text{ for some } v \in \mathbb{R}^n\}.$$

Show that the faces F_v have codimension n. Compare with Exercise 1 above.

13. Let $K \subset \mathbb{R}^n$ be a compact convex set with the origin not in K. Show that the conical hull of K, cone(K) is closed. Construct an explicit example that shows that the condition $0 \notin K$ is necessary.

14. Show that $P_{n,2d}$ and $\Sigma_{n,2d}$ are closed full-dimensional cones with no lines in $\mathbb{R}[x]_{2d}$. (Hint: Use Exercise 13 for closure of $\Sigma_{n,2d}$).

15. Show that a univariate nonnegative polynomial is a sum of at most 2 squares. What can be said about polynomials nonnegative on an interval (not necessarily bounded)?

Quadratic Persistence and Pythagoras Number.

16. Let X be a projective variety, with vanishing graded ideal $I = I(X)$. Let v be a point of X and X_v be the projection of X away from v. Show that

$$\dim I(X_2) - \dim I(X_v)_2 \leq \operatorname{codim} X,$$

i.e. if we project away from a point the ideal loses at most $\operatorname{codim} X$ quadrics.

17. Using notation of Exercise 16 and v a real point of X, show that

$$P_{X_v} \subset P_X(v).$$

18. Show the following upper bound on Pythagoras number of a variety X (not necessarily irreducible):

$$\binom{\Pi(X)+1}{2} \leq \dim R_2.$$

19. Determine quadratic persistence of a toric variety corresponding to $a \times b$ lattice box.

20. Determine quadratic persistence of Veronese embeddings of $\mathbb{P}^n$.

Duaiity and Moment Problems.

21. Write down a semidefinite program for showing that $5+6x-4x^2-4x^3+2x^4$ is nonnegative on $\mathbb{R}$. Solve it without using an SDP solver.

22. Let $C = L \cap \mathbb{S}^n_+$ be a spectrahedral cone. Show that A spans an extreme ray of C if and only if $\ker A$ is maximal (by inclusion) among all matrices in L. More precisely if $B \in L$ and $\ker A \subseteq \ker B$ then $B = \lambda A$ for some $\lambda \in \mathbb{R}$.

23. Let X be a projective variety. For $\ell \in \Sigma^*_X$ the kernel W of the associated quadratic form Q_ℓ is a subspace of linear forms in R_1, whose vanishing defines a linear subspace V, Use exercise 22 to show that if ℓ spans an extreme ray of $\mathbb{S}^n_+$ then $V \cap X(\mathbb{C}) = \emptyset$. Conclude

that $\dim W \geq \dim X + 1$.

24. Let X be a variety of minimal degree. Then it is known that X is arithmetically Cohen-Macaulay and $\dim R_2 = \binom{\operatorname{codim} X+1}{2}$. Use these facts and Exercise 23 to show that $\Sigma_X^* = P_X^*$ and therefore $\Sigma_X = P_X$.

25. Let $\mathbb{R}[x]_{\leq d}$ denote the vector space of univariate polynomials of degree at most d. Describe all linear functionals $\ell \in \mathbb{R}[x]_{\leq 4}^*$ with positive semidefinite moment matrix, but no representing measure.