Fall School, Algebraic Geometry, Varieties, Polyhedra and Computation

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Exercise Sheet - Tensor Rank and Complexity - Giorgio Ottaviani

Exercise 1. A tensor $t \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ defines three contractions

$(\mathbb{C}^a)^\vee \to \mathbb{C}^b \otimes \mathbb{C}^c$

$(\mathbb{C}^b)^\vee \to \mathbb{C}^a \otimes \mathbb{C}^c$

$(\mathbb{C}^c)^\vee \to \mathbb{C}^a \otimes \mathbb{C}^b$

and we call their ranks respectively $r_1(t)$, $r_2(t)$, $r_3(t)$, they are called multilinear ranks.

- (a) Prove that

$$\mathrm{rk}(t) = 1 \iff \begin{cases} r_1(t) = 1 \\ r_2(t) = 1 \\ r_3(t) = 1 \end{cases}$$

- (b) Prove that $(r_1, r_2, r_3) = (1, 2, 2)$ is admissible, while $(r_1, r_2, r_3) = (1, 1, 2)$ is not allowed.
- (c) Prove that $r_i \leq \mathrm{brk}(t) \leq \mathrm{rk}(t) \leq r_j r_k \ \forall i, j, k$ (brk is the border rank).
- (d) Deduce that $r_1 \leq r_2 r_3$, $r_2 \leq r_1 r_3$, $r_3 \leq r_1 r_2$.

Exercise 2. (Geometric version) Let A, B, C be three complex vector spaces of dimension two.

- a)Prove that, in the language of Exercise 1, the only admissible triples for (r_1, r_2, r_3) are $(1, 1, 1)$, $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$, $(2, 2, 2)$.
- b) In the matrix space $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ the variety of matrices of rank one is a smooth quadric surface Q. Interpretate the three contractions $(\mathbb{C}^2)^\vee \to \mathbb{C}^2 \otimes \mathbb{C}^2$ as three pencils of matrices $\mathbb{P}^1 \to \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ and characterize them depending on the intersection of the pencil with Q, according to each admissible triple. For example in the case $(1, 1, 1)$ the contraction maps $\mathbb{P}^1 \to \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ are degenerate, the image collapse to a single point contained in Q. In the case $(1, 2, 2)$ there are two contraction maps where the image is a line all contained in Q. Go ahead with a complete description of all cases.
- c)Prove that the cases $(1, 1, 1)$, $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 1)$ correspond to a unique orbit, while the case $(2, 2, 2)$ splits into two orbits.*Hint: the pencil may be transversal or tangent to* Q.
- d) Describe the graph of the six orbits in $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ and the graph of the seven orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ (they include the zero orbit). A reference is [GKZ, Example 14.4.5].

Exercise 2'. (This is the Algebraic Version of Exercise 2, both languages are useful) Let A, B, C be three vector spaces of dimension two, with basis respectively given by $\{a_0, a_1\}$, $\{b_0, b_1\}$, $\{c_0, c_1\}$.

(1) For any tensor $t = \sum_{i,j,k=0,\ldots 1} t_{ijk} a_i b_j c_k$, write it as a 2×2 with coefficients linear in c_i, so as a *pencil* of 2×2 matrices.

(2) Compute the condition that the previous matrix is singular, it is a quadratic equation in c_i giving a pair of points.

(3) Compute the condition that the previous pair of points consists of a double point, get a polynomial of degree 4 in t_{ijk}, which is called the hyperdeterminant of t and we denote as $Det(t)$. It corresponds to a pencil tangent to Q, in the geometric language of Exercise 2.

(4) Show that the above pair gives the two summands of t, when t has rank two. Prove that in the dense orbit, over $\mathbb{C}$, there is a unique decomposition as a sum of two decomposable tensors (this was the main result by C. Segre).

(5) In the real case, prove that the sign of $Det(t)$ allows to detect if a real $2 \times 2 \times 2$ tensor has rank 2 or 3.

(6) Prove that $w = a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0$ has (complex) rank 3. This is called a W-state in Quantum Information Theory. Write infinitely many decompositions of w as the sum of three decomposable tensors. *Hint: in the last summand you could modify with* $(a_0 \sin\theta + a_1 \cos\theta)(b_0 \cos\theta + b_1 \sin\theta)c_0$

(7) Prove that $a_0b_0c_0 + a_0b_1c_1$ has infinitely many decompositions. Which are its multilinear ranks (r_1, r_2, r_3)? *Hint: in the first summand you could modify with* $a_0(b_0 \cos\theta + b_1 \sin\theta)(c_0 \cos\theta + c_1 \sin\theta)$

Exercise 3.

(a) The following $2 \times 2 \times 2$ tensors t_1, t_2, t_3 fill the first column of the following table. Which is which ? Can you decompose them ?

t	$\mathrm{rk}_{\mathbb{R}}(t)$	$\mathrm{rk}_{\mathbb{C}}(t)$
?	2	2
?	3	2
?	3	3

$t_1 = 4a_0b_0c_0 + 2a_1b_0c_0 - a_0b_1c_0 + 2a_0b_0c_1$

$t_2 = a_0b_0c_0 + 2a_1b_1c_0 + 3a_1b_0c_1 + 6a_0b_1c_1$

$t_3 = a_0b_0c_0 - 2a_1b_1c_0 - 3a_1b_0c_1 - 6a_0b_1c_1$

- (b)Write down the finite number of orbits for the action of $GL(2,\mathbb{R})\times GL(2,\mathbb{R})\times GL(2,\mathbb{R})$ on $\mathbb{R}^2\otimes\mathbb{R}^2\otimes\mathbb{R}^2$.

Exercise 4. (Harder) In [L, Theorem 10.10.2.6] it is reported that the maximal rank for tensors in $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ is 4. The proof by Brylinski examines the rank of the map $\mathbb{P}^1\to\mathbb{P}(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$. Try to repeat the argument for tensors in $\mathbb{R}^2\otimes\mathbb{R}^2\otimes\mathbb{R}^2\otimes\mathbb{R}^2$ by using Exercise 3.

Exercise 5. (Harder) A rank in real tensor space is called typical if it is attained in a set with nonempty interior (equivalently, with positive volume). It is known that the typical ranks of $\mathbb{R}^2\otimes\mathbb{R}^2\otimes\mathbb{R}^2$ are 2 and 3.
What are the typical ranks for $\mathbb{R}^2\otimes\mathbb{R}^2\otimes\mathbb{R}^2\otimes\mathbb{R}^2$?

References

[GKZ] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky: *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[L] J. M. Landsberg, *Tensors: Geometry and Applications*, AMS, Providence, 2012.