Moduli of Differentials and Teichmüller Dynamics – Exercises

Exercise 1. Find $\mathcal{H}(\mu)$ that contains each of the following translation surfaces:

(1) A decagon of type $a+b+c+d+e=b+a+d+c+e$.
(2) An $2n$-gon of type $v_1+\cdots+v_n=v_n+\cdots+v_1$.
(3) A big flat torus minus a small flat torus.
(4) The Euclidean plane (with a point at ∞) minus a flat torus.
(5) A pillow case.
(6) The surface of a cube.

Remark. (4), (5), (6) are generalizations of holomorphic one-forms.

Exercise 2. Prove that $\mathcal{H}(4)$ has exactly two connected components.

Exercise 3. Draw a translation surface in $\mathcal{H}(2)$ and compute the Arf invariant.

Exercise 4. A variety is called unirational if it can be dominated by a projective space. Prove that each of the following strata is unirational.

(1) $\mathcal{H}(2g-2)^{\mathrm{hyp}}$.
(2) All strata (components) in genus 3.

Exercise 5. Prove that for any translation surface in $\mathcal{H}(3,1)$, the underlying Riemann surface is not hyperelliptic.

Exercise 6. Suppose $(X,\omega)\in\mathcal{H}(2,1,1)$ with $(\omega)_0=2p_1+p_2+p_3$. Prove that $h^0(X,p_1+p_2)=1$.

Exercise 7. Prove that the projection of any $\mathrm{GL}_2^+(\mathbb{R})$-orbit to the moduli space of genus g curves $\mathcal{M}_g$ factors through the upper-half plane $\mathbb{H}$.

Exercise 8. Show that $(X,\omega)\in\mathcal{H}(\mu)$ corresponds to a square-tiled surface if and only if all period coordinates of (X,ω) belong to $\mathbb{Z}\oplus\mathbb{Z}i$, namely, if and only if (X,ω) is an integral point in $\mathcal{H}(\mu)$ under the period coordinates.

Exercise 9. Let $\Delta=\cup_{i=0}^{[g/2]}\Delta_i$ be the total boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, where a general point in the boundary component Δ_i parameterizes a nodal union of a genus i curve and a genus $g-i$ curve for $i>0$ and a general point in Δ_0 parameterizes an irreducible nodal curve of geometric genus $g-1$. Prove that for a Teichmüller curve in $\mathcal{M}_g$ generated by a square-tiled surface, its closure in $\overline{\mathcal{M}}_g$ does not intersect Δ_i for any $i>0$.

Exercise 10. Suppose a family of translation surfaces in $\mathcal{H}(2)$ degenerate to two elliptic curves E and E' union at a node q. Moreover, suppose the limit of the double zeros is a point $p \in E \setminus q$. Prove that $2p \sim 2q$ in E.

Exercise 11. Show that in Exercise 1, Part (2), the underlying Riemann surface is hyperelliptic. Moreover, find all of its Weierstrass points.

Exercise 12. Let X be a hyperelliptic Riemann surface defined by the equation

$$x^2 = (z - a_1) \cdots (z - a_{2g+2})$$

(extending to ∞), where $a_1, \ldots, a_{2g+2}$ are fixed distinct points in $\mathbb{C}$ and the double cover $X \to \mathbb{P}^1$ is given by $(x, z) \mapsto z$. Prove that the following one-forms are holomorphic (hence they form a basis of the space of holomorphic one-forms on X):

$$\frac{dz}{x}, \ \frac{zdz}{x}, \ldots, \frac{z^{g-1}dz}{x}.$$

Exercise 13. Draw two translation surfaces in $\mathcal{H}(4)$ such that their Arf invariants are different.