Moduli of Differentials and Teichmüller Dynamics
– Exercises

Exercise 1. Find $\mathcal{H}(\mu)$ that contains each of the following translation surfaces:

1. A decagon of type $a+b+c+d+e = b+a+d+c+e$.
2. An $2n$-gon of type $v_1 + \cdots + v_n = v_n + \cdots + v_1$.
3. A big flat torus minus a small flat torus.
4. The Euclidean plane (with a point at $\infty$) minus a flat torus.
5. A pillow case.
6. The surface of a cube.

Remark. (4), (5), (6) are generalizations of holomorphic one-forms.

Exercise 2. Prove that $\mathcal{H}(4)$ has exactly two connected components.

Exercise 3. Draw a translation surface in $\mathcal{H}(2)$ and compute the Arf invariant.

Exercise 4. A variety is called unirational if it can be dominated by a projective space. Prove that each of the following strata is unirational.

1. $\mathcal{H}(2g - 2)^{hyp}$.
2. All strata (components) in genus 3.

Exercise 5. Prove that for any translation surface in $\mathcal{H}(3,1)$, the underlying Riemann surface is not hyperelliptic.

Exercise 6. Suppose $(X, \omega) \in \mathcal{H}(2,1,1)$ with $(\omega)_0 = 2p_1 + p_2 + p_3$. Prove that $h^0(X, p_1 + p_2) = 1$.

Exercise 7. Prove that the projection of any $\text{GL}_2^+(\mathbb{R})$-orbit to the moduli space of genus $g$ curves $\mathcal{M}_g$ factors through the upper-half plane $\mathbb{H}$.

Exercise 8. Show that $(X, \omega) \in \mathcal{H}(\mu)$ corresponds to a square-tiled surface if and only if all period coordinates of $(X, \omega)$ belong to $\mathbb{Z} \oplus \mathbb{Z}i$, namely, if and only if $(X, \omega)$ is an integral point in $\mathcal{H}(\mu)$ under the period coordinates.

Exercise 9. Let $\Delta = \bigcup_{i=0}^{[g/2]} \Delta_i$ be the total boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, where a general point in the boundary component $\Delta_i$ parameterizes a nodal union of a genus $i$ curve and a genus $g-i$ curve for $i > 0$ and a general point in $\Delta_0$ parameterizes an irreducible nodal curve of geometric genus $g-1$. Prove that for a Teichmüller curve in $\mathcal{M}_g$ generated by a square-tiled surface, its closure in $\overline{\mathcal{M}}_g$ does not intersect $\Delta_i$ for any $i > 0$. 
Exercise 10. Suppose a family of translation surfaces in $\mathcal{H}(2)$ degenerate to two elliptic curves $E$ and $E'$ union at a node $q$. Moreover, suppose the limit of the double zeros is a point $p \in E \setminus q$. Prove that $2p \sim 2q$ in $E$.

Exercise 11. Show that in Exercise 1, Part (2), the underlying Riemann surface is hyperelliptic. Moreover, find all of its Weierstrass points.

Exercise 12. Let $X$ be a hyperelliptic Riemann surface defined by the equation

$$x^2 = (z - a_1) \cdots (z - a_{2g+2})$$

( extending to $\infty$), where $a_1, \ldots, a_{2g+2}$ are fixed distinct points in $\mathbb{C}$ and the double cover $X \to \mathbb{P}^1$ is given by $(x, z) \mapsto z$. Prove that the following one-forms are holomorphic (hence they form a basis of the space of holomorphic one-forms on $X$):

$$\frac{dz}{x}, \frac{zdz}{x}, \ldots, \frac{z^{g-1}dz}{x}.$$

Exercise 13. Draw two translation surfaces in $\mathcal{H}(4)$ such that their Arf invariants are different.