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Kombinatorische Kommutative Algebra
Sommersemester 2007

Homework 7.

June 7, 2007

Due June 14, 2007

Exercise 1.

Dualize the hull resolution from exercise 5.3 to describe a free resolution of the ideal

$$I = \langle x^3, y^4, z^4 \rangle \cap \langle x, y^2, z \rangle \cap \langle x^3, z^2 \rangle \cap \langle y^3, z^2 \rangle \cap \langle x^4, y^3 \rangle \cap \langle x^3, y^4 \rangle \ .$$

Exercise 2.

In the context of the previous exercise, illustrate the objects from the proof of the resolution duality theorem for $\mathbf{b} = (4, 4, 1)$: identify the U_i and their nerve $\mathcal{N}$.

Exercise 3.

Let $a, b \in \mathbb{Z}^2$ linearly independent. Prove that a, b is a lattice basis of $\mathbb{Z}^2$, if $0, a, b$ are the only lattice points in the triangle with vertices $0, a, b$.

(*Hint:* Why are there also in the parallelogram with vertices $0, a, b, a + b$ no lattice points except the four vertices?)

Exercise 4.

Say a matrix is *integral*, if all entries are in $\mathbb{Z}$. Let $\mathrm{GL}_n(\mathbb{Z})$ be the set of integral $n \times n$-matrices G that are *invertible*, i.e., there is an integral $n \times n$-matrix G' such that $GG' = \mathrm{id}$. Show that an integral $n \times n$-matrix G is invertible if and only if the determinant is ± 1.

(*Hint:* Use a well-known Linear Algebra formula for the adjoint matrix.)

Extrablatt: Nerve Lemma (optional)

Exercise 1.

Suppose $X = X' \cup X''$ are simplicial complexes. Show that there is a short exact sequence of cochain complexes

$$0 \longleftarrow C^\bullet(X' \cap X'') \longleftarrow C^\bullet(X') \oplus C^\bullet(X'') \longleftarrow C^\bullet(X) \longleftarrow 0$$

Exercise 2.

Suppose X and X' are simplicial complexes on vertex sets $[n]$ and $[n']$, respectively. A map $\phi\colon [n] \to [n']$ is simplicial $X \to X'$ if $\phi(\sigma) \in X'$ for all $\sigma \in X$.

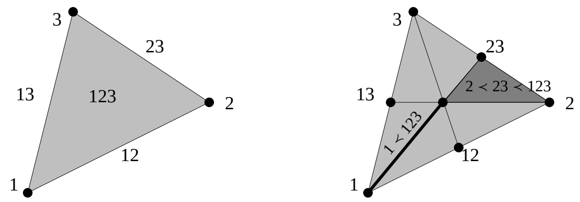
Show that the induced linear maps $\phi^k\colon C^k(X') \to C^k(X)$ are cochain maps, i.e., $\delta\phi = \phi\delta$. Conclude that there are induced homomorphisms $\phi^*\colon \widetilde{H}^k(X') \to \widetilde{H}^k(X)$.

Exercise 3.

The barycentric subdivision of a simplicial complex X is a simplicial complex $\mathrm{bsd}(X)$ on the vertex X whose simplicies are the chains:

$$\{\sigma_1, \ldots, \sigma_r\} \in \mathrm{bsd}(X) \quad \text{if and only if} \quad \sigma_1 \prec \ldots \prec \sigma_r$$

(after reindexing if necessary).



Let $X = \bigcup_{i=1}^N X_i$ be simplicial complexes on $[n]$, and denote the nerve by $\mathcal{N} = \{I \subset [N] \ :\ \bigcap_{i\in I} X_i \neq \emptyset\}$.

Show that the map $\sigma \mapsto \min\{\, i \ :\ \sigma \in X_i \,\}$ is simplicial $\mathrm{bsd}(X) \to \mathcal{N}$.

Exercise 4.

Let $X = \bigcup_{i=1}^N X_i$ be simplicial complexes on $[n]$, so that $\widetilde{H}^k(\bigcap_{i\in I} X_i) = 0$ for $k \geq 0$ and all $I \subseteq [N]$. Denote the nerve by $\mathcal{N}$. Set $X' := \bigcup_{i=1}^{N-1} X_i$ with nerve $\mathcal{N}'$, and $Y := X' \cap X_N = \bigcup_{i=1}^{N-1} X_i \cap X_N$ with nerve $\mathcal{N}_0$.

Show that there are commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
\widetilde{H}^k(X') & \longrightarrow & \widetilde{H}^k(Y) & \longrightarrow & \widetilde{H}^{k+1}(X) & \longrightarrow & \widetilde{H}^{k+1}(X') & \longrightarrow & \widetilde{H}^{k+1}(Y) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{H}^k(\mathcal{N}') & \longrightarrow & \widetilde{H}^k(\mathcal{N}_0) & \longrightarrow & \widetilde{H}^{k+1}(\mathcal{N}) & \longrightarrow & \widetilde{H}^{k+1}(\mathcal{N}') & \longrightarrow & \widetilde{H}^{k+1}(\mathcal{N}_0)
\end{array}$$

Use induction on N to deduce that $\widetilde{H}^k(X) \cong \widetilde{H}^k(\mathcal{N})$.