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Kombinatorische Kommutative Algebra
Sommersemester 2007

Homework 8.

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Due June 21, 2007

Exercise 1.

Let $a_1, \ldots, a_n \in \mathbb{N}_{\geq 1}$ with greatest common divisor 1. Then there are only finitely many natural numbers not contained in the semigroup $\mathbb{N}\{a_1, \ldots, a_n\}$. Prove this for $n = 2$ (Bonuspoint for general n).

(*Hint:* For $n = 2$ use and prove the fact that $a_2 + a_1\mathbb{Z}$ generates $\mathbb{Z}/a_1\mathbb{Z}$.)

Exercise 2.

Let A be a finitely generated free abelian group, R an A-graded commutative ring, and $I \subsetneq R$ a homogeneous ideal. Show that I is a prime ideal if and only if the following implication holds for any $a \in A$: For $r, s \in R_a$ with $rs \in I$ it follows $r \in I$ or $s \in I$.

(*Hint:* Since $A \cong \mathbb{Z}^d$, one can use a suitable total ordering on A to define a leading term of an element in R. Then do induction.)

Exercise 3.

1. (*Smith normal form*) Let M be an integral $n \times m$-matrix. Then there exist matrices $U \in \mathrm{GL_n}(\mathbb{Z})$ and $V \in \mathrm{GL_m}(\mathbb{Z})$ such that UMV is a diagonal $n \times m$-matrix with entries $d_1, \ldots, d_s$ on the diagonal (called *Elementarteiler*) for $s := \min(n, m)$, where d_i divides d_{i+1} for $i = 1, \ldots, s - 1$.

Give a proof of this statement in the case $m = n = 2$.

(*Hint:* Show that it is possible by subtracting a row (respectively, column) from another row (respectively, column) to finally get a matrix where the upper left entry is the greatest common divisor of all entries.)

2. Application: Given generators $l_1, \ldots, l_m \in \mathbb{Z}^n$ of a lattice $L \subseteq \mathbb{Z}^n$. Form the $n \times m$-matrix $\mathcal{L}$ with columns $l_1, \ldots, l_m$. Show that $A := \mathbb{Z}^n/L$ is isomorphic to $\mathbb{Z}^{n-s} \oplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z}$ (in the notation of 1. for $M := \mathcal{L}$). Moreover, if A is free abelian of rank d, then $d = n - s$, and the $\mathbb{Z}$-linear map $\mathbb{Z}^n \to A \cong \mathbb{Z}^d$ is given by the $d \times n$-matrix consisting of the last d rows of U.

Exercise 4.

Let $L :=< (-1,3,5,3),(3,-2,-8,-2) \subseteq \mathbb{Z}^4$. Does this define an affine semigroup (in $\mathbb{Z}^4/L$)? Calculate the lattice ideal I_L.

(*Hint:* In Maple you may want to use the following commands:
with(linalg): ismith($\mathcal{L}$); with(PolynomialIdeals): Saturate($I_{\mathcal{L}}$);)