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Kombinatorische Kommutative Algebra
Sommersemester 2007

Homework 9.

June 21, 2007

Due June 28, 2007

Exercise 1.

Let C be a d-dimensional (finitely generated, polyhedral, rational) cone in $\mathbb{R}^d$. Show that C is pointed if and only if the dual cone $C^\vee$ has also dimension d.

(*Hint:* You may use duality $(C^\vee)^\vee = C$, and the facts that the boundary of C (respectively, $C^\vee$) is covered by the facets of C (respectively, $C^\vee$), and that $C^\vee$ is generated by the outer normals of the facets of C.)

Exercise 2.

Let C as in the previous exercise. Show that C contains a lattice basis of $\mathbb{Z}^d$.

(*Hint:* Induction on the dimension, you may also use all facts in the hint of Exercise 1.)

Exercise 3.

Let C be the cone in $\mathbb{R}^4$ generated by $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(1, 2, 3, 5)$. Show that $(1, 1, 1, 1)$ is contained in the Hilbert basis of $C \cap \mathbb{Z}^4$.

Exercise 4.

Let M be an integral $n \times m$-matrix. How can one compute a lattice basis of the kernel of M using Smith normal form? As an example, calculate a basis of the lattice of integer solutions to the equation $2u_1 + 7u_2 = 3u_3 + 5u_4$.

(Bonuspoint: Calculate the Hilbert basis of the affine semigroup of all nonnegative integer solutions (e.g., via the Lawrence ideal).)