Problems from the Cottonwood Room.

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Abstract. This collection was compiled by Christian Haase and Bruce Reznick from problems presented at the problem sessions, and submissions solicited from the participants of the AMS/IMS/SIAM summer Research Conference on Integer points in polyhedra.

Contents

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Questions about Ehrhart coefficients.

Presented by Matthias Beck (San Francisco State University).

Given a rational d-polytope P, Ehrhart's Theorem (see, e.g., [**3**]) states that the lattice-point enumerating function $i_P(t) := \#\left(tP \cap \mathbb{Z}^d\right)$ for $t \in \mathbb{N}$ is a *quasipolynomial*, that is, a function of the form

$$c_d(t)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_0(t),$$

where $c_0, \ldots, c_d$ are periodic functions.

Conjecture 1.1 (Thomas Zaslavsky, unpublished). *For the Ehrhart quasipolynomial* $i_P(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_0(t)$*, denote the period of* c_k *by* p_k*. Then* $1 = p_d \le p_{d-1} \le \cdots \le p_0$*, or maybe even* $1 = p_d \mid p_{d-1} | \cdots \mid p_0$*.*

This conjecture was disproved during the conference independently by David Einstein (unpublished) and Tyrrell McAllister and Kevin Woods [**4**].

Problem 1.2. *Characterize those polytopes for which Zaslavsky's Conjecture is true.*

For the remaining problems, P will be a lattice polytope: the vertices have integral coordinates. In this case, i_P is an honest polynomial, i.e., $p_d = \cdots = p_0 = 1$.

Problem 1.3. *There are lattice (3-)polytopes whose Ehrhart polynomials have negative coefficients (see, e.g.,* [**5**, Example 3.5]*). Characterize those polytopes whose Ehrhart polynomials do not have negative coefficients.*

Problem 1.4. *The roots of the Ehrhart polynomial of the cross polytope*

$$\left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : \ |x_1| + \cdots + |x_d| \le 1\right\}$$

all have real parts equal to $-1/2$ [**2, 6**]*. Find other polytopes whose Ehrhart polynomials exhibit such special behavior.*

Problem 1.5. *Improve the following bounds (proved in* [**1**]*):*

(i) *The roots of Ehrhart polynomials of lattice d-polytopes are bounded above in norm by* $1 + (d+1)!$ *.*
(ii) *All real roots of Ehrhart polynomials of d-dimensional lattice polytopes lie in the half-open interval* $[-d, \lfloor d/2 \rfloor)$*.*

Remark 1.1.

(i) For $d = 4$, this upper bound can be improved to 1 (which is sharp).
(ii) We thought for a while that the upper bound in (b) could be conjectured to be 1, but then found a counterexample (a class of 0/1 order polytopes, described in [**1**]). Incidentally, the same counterexample got rid of another conjecture, namely that the Ehrhart polynomial of any 0/1 polytope has only positive coefficients.

Conjecture 1.6 (Beck-DeLoera-Develin-Pfeifle-Stanley). *All roots* α *of Ehrhart polynomials of lattice d-polytopes satisfy* $-d \le \operatorname{Re} \alpha \le d - 1$*.*

CONJECTURE 1.7 (Jesús DeLoera). *For the cyclic polytope $C(n,d)$ realized with integral vertices on the moment curve $\nu_d(t) := \left(t, t^2, \ldots, t^d\right)$,*

$$i_{C(n,d)}(m) = \mathrm{vol}(C(n,d))\, m^d + i_{C(n,d-1)}(m).$$

Equivalently,

$$i_{C(n,d)}(m) = \sum_{k=0}^{d} \mathrm{vol}_k(C(n,k))\, m^k.$$

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A conjecture about the Euler characteristic of algebraic varieties over the fields $\mathbb{F}_q$, $\mathbb{R}$, $\mathbb{C}$ and the division ring $\mathbb{H}$.

Presented by BEIFANG CHEN (Hong Kong University).

Let X be an algebraic variety (either affine or projective) whose defining equations have integral coefficients. Then X is well-defined over the finite field $\mathbb{F}_q$ of q elements, the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, and the division ring $\mathbb{H}$ of quaternions. We denote the corresponding varieties by $X_{\mathbb{F}_q}$, $X_{\mathbb{R}}$, $X_{\mathbb{C}}$, and $X_{\mathbb{H}}$, respectively. Let $f(X,q)$ be the number of elements of the finite set $X_{\mathbb{F}_q}$. In many cases, the function $f(X,q)$ turns out to be a polynomial function of q. For instance, for the projective variety $\mathrm{Gr}(n,k)$, the Grassmanian of k-subspaces of the n-dimensional vector space $\mathbb{F}_q^n$, the function $f(\mathrm{Gr}(n,k),q)$ is the Gaussian polynomial

$$\binom{n}{k}_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})},$$

the q-analog of the binomial coefficient $\binom{n}{k}$; see [**4**].

CONJECTURE 2.1. *Let X be a smooth projective variety whose defining equations have integral coefficients. If $f(X,q)$ is a polynomial function of q, then*

$$\chi(X_{\mathbb{F}}) = f(X, \chi(\mathbb{F})),$$

where $\chi(X_{\mathbb{F}})$ is the Euler characteristic of the variety $X_{\mathbb{F}}$ over the field $\mathbb{F}$, and $\chi(\mathbb{F})$ is defined by

$$\chi(\mathbb{F}) = \begin{cases} q & \text{if } \mathbb{F} = \mathbb{F}_q \\ -1 & \text{if } \mathbb{F} = \mathbb{R} \\ 1 & \text{if } \mathbb{F} = \mathbb{C}, \mathbb{H}. \end{cases}$$

The conjecture was verified by the author for some special cases such as Grassmanians, partition varieties [**2**], quiver varieties, and toric varieties, etc. In particular, for the complex projective variety $X_{\mathbb{C}}$, the formula $\chi(X_{\mathbb{C}}) = f(X, 1)$ can be derived from the Weil conjecture [**3**].

Of course there is no need to restrict ourselves to projective varieties and to assume the smoothness condition in the conjecture. The same conjecture can be formulated for affine and singular varieties as long as $f(X, q)$ is a polynomial function. However, if X is an affine variety whose defining equations have integral coefficients, then the Euler characteristic $\chi(X_{\mathbb{F}})$ should be understood as the combinatorial Euler characteristic defined in [**1**], the alternating sum of the number of cells in a triangulation of $X_{\mathbb{F}}$.

One may only assume that $f(X, q)$ is a polynomial function for $q = p^k$, where p is a fixed prime. One may also consider solutions over $\mathbb{Z}$ and divide the integers into some classes. For each of these classes we obtain some equations to define a variety. The varieties obtained in this way correspond to some branches of the variety $X_{\mathbb{R}}$ over the real field $\mathbb{R}$. The conjecture can be similarly formulated for each of these classes.

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Lattice points in homogeneously expanding compact domains.

Presented by Lenny Fukshansky (Texas A&M University).

A variety of interesting and important questions in geometric combinatorics and in geometry of numbers is connected to counting integer lattice points in compact subsets of a Euclidean space. In case such a subset is a rational polyhedron, the problem can be reformulated in terms of Ehrhart polynomial. In a general situation, however, one often has to rely on estimates of asymptotic nature. A good example of such an estimate is presented by S. Lang in Theorem 2 on p. 128 of [**3**]. We state it here. In the rest of this note we assume that $N > 1$ is an integer.

Theorem 3.1 ([**3**]). *Let D be a compact subset of $\mathbb{R}^N$, and let L be a lattice of rank N in $\mathbb{R}^N$ with fundamental domain F. Assume that the boundary ∂D of D is Lipschitz-parametrizable. Then for each positive $t \in \mathbb{R}$ the number of points of L in tD is given by the following asymptotic formula:*

$$|L \cap tD| = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(F)} t^N + O(t^{N-1}), \tag{3.1}$$

where Vol *stands for volume in $\mathbb{R}^N$, and the constant in O depends on L, N, and Lipschitz constants.*

We recall that the condition that ∂D is *Lipschitz-parametrizable* means that there exists a finite collection of maps $\varphi_j : [0, 1]^N \longrightarrow \partial D$, the union of images of which covers ∂D and there exists a constant K such that for all $\boldsymbol{x}, \boldsymbol{z} \in [0,1]^N$

$$|\varphi_j(\boldsymbol{x}) - \varphi_j(\boldsymbol{z})| \leq K|\boldsymbol{x} - \boldsymbol{z}|,$$

for each j, where $|\ |$ stands for the sup-norm on $\mathbb{R}^N$, i.e. $|\boldsymbol{x}| = \max_{1 \leq i \leq N} |x_i|$. The constant K is called the associated *Lipschitz constant*.

Notice that the main term in the upper bound in (3.1) is explicit and easily computable, but the error term is implicit. Loosely speaking, the main term of such an asymptotic estimate counts the number of "interior points" of L in D, i.e. points that are away from the boundary, and the error term accounts for the points near the boundary. For practical applications it is important to be able to explicitly estimate the error term. Such an estimate was carried out for instance by H. Davenport in [**1**] (see also [**5**] for a very nice account and generalizations of Davenport's theorem). However Davenport's bound on the error term depends on projection volumes of D onto certain subspaces of $\mathbb{R}^N$ as well as determinants of projections of L onto these subspaces. These are hard to compute. In some situations one would prefer perhaps cruder, but more tractable bounds on the error term. An alternative approach to this problem is to try to "quantify" the original argument in Lang's theorem. This has been partially done by P. G. Spain in [**4**]. We briefly outline this approach here and ask some further questions.

We start by sketching out the main idea of proof of Theorem 3.1. One proceeds by noticing that for each positive real number t

$$m(t) \leq |L \cap tD| \leq m(t) + b(t),$$

where

(i) $m(t)$ = number of $\boldsymbol{x} \in L$ such that $F + \boldsymbol{x}$ belongs to the interior of of tD,
(ii) $b(t)$ = number of $\boldsymbol{x} \in L$ such that $F + \boldsymbol{x}$ intersects $\partial(tD)$.

It is obvious that

$$m(t) \leq \frac{\mathrm{Vol}(tD)}{\mathrm{Vol}(F)} = \frac{\mathrm{Vol}(D)}{\mathrm{Vol}(F)} t^N,$$

which produces the main term. In order to produce the error term one needs to estimate $b(t)$. This unfortunately is not so easy. Lang only proves that $b(t) = O(t^{N-1})$ using the fact that the boundary $\partial(tD)$ of tD is Lipschitz-parametrizable, but does not exhibit any explicit upper bound. Although, as we discussed above, there are other methods for estimating the error term, the quantity $b(t)$ seems to be interesting in its own right. It can, for instance, be related to a covering problem, namely: how many translates of the closure of the fundamental domain F does it take to cover the the compact domain tD? Such a number can again be approximated by the expression $m(t) + b(t)$ as above. The following estimate for $b(t)$ in the special case when $L = \mathbb{Z}^N$ was produced by P G. Spain.

THEOREM 3.2 ([**4**]). *Let D be as in Theorem 3.1, so that the boundary ∂D is Lipschitz-parametrizable with Lipschitz constant K. Let $L = \mathbb{Z}^N$. Let F be the fundamental domain of L with respect to the standard basis, and let $b(t)$ as above be the number of translates of F that have nonempty intersection with $\partial(tD)$ where $t \geq 1/K$. Then*

$$b(t) \leq 2^N (Kt + 1)^{N-1}, \tag{3.2}$$

and therefore

$$|\mathbb{Z}^N \cap tD| \leq \mathrm{Vol}(D)t^N + 2^N(Kt+1)^{N-1} = \mathrm{Vol}(D)t^N + 2^N \sum_{i=0}^{N-1} K^i t^i. \tag{3.3}$$

PROBLEM 3.1. *Provide an explicit bound on $b(t)$ as in Theorem 3.2 for a general lattice L.*

Perhaps one can modify Spain's argument to produce a solution to Problem 3.1. Notice that the upper bound on $|L \cap tD|$ as it comes out in (3.3) is a polynomial in t whose coefficients depend on volume of D and on the Lipschitz constant K. This suggests a certain analogy with Ehrhart polynomial: one may look for polynomial upper and lower bounds on $|L \cap tD|$ for some more or less general instances of L and D. Here is an example for a simple choice of D when L is any sublattice of $\mathbb{Z}^N$. Let

$$C_t^N = \{\boldsymbol{y} \in \mathbb{R}^N : \max\{|y_1|, ..., |y_N|\} \leq t\},$$

that is, C_t^N is a cube with side length $2t$ centered at the origin in $\mathbb{R}^N$.

THEOREM 3.3 ([**2**]). *Let $\Lambda \subseteq \mathbb{Z}^N$ be a lattice of full rank in $\mathbb{R}^N$ of determinant Δ. Then for each point $\boldsymbol{z}$ in $\mathbb{R}^N$ we have*

$$\begin{aligned} \left(\frac{2^N}{\Delta}\right) t^N &\leq |\Lambda \cap (C_t^N + \boldsymbol{z})| \\ &\leq \left(\frac{2t}{\Delta} + 1\right)(2t+1)^{N-1} = \left(\frac{2^N}{\Delta}\right) t^N + \sum_{i=1}^{N-1} 2^i \left(\frac{1}{\Delta} + 1\right) t^i + 1. \end{aligned} \tag{3.4}$$

In [**2**] an analogous bound for a rectangular box instead of a cube is also produced; the result of Theorem 3.3 is extended to lattices of not full rank and to certain modules over the ring of algebraic integers in a number field, viewed as $\mathbb{Z}$-modules.

PROBLEM 3.2. *Produce explicit polynomial bounds like (4) for more general choices of compact domain D.*

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Reflexive polytopes in dimension 2 and 3, and the numbers 12 and 24.

Presented by CHRISTIAN HAASE (Duke University).

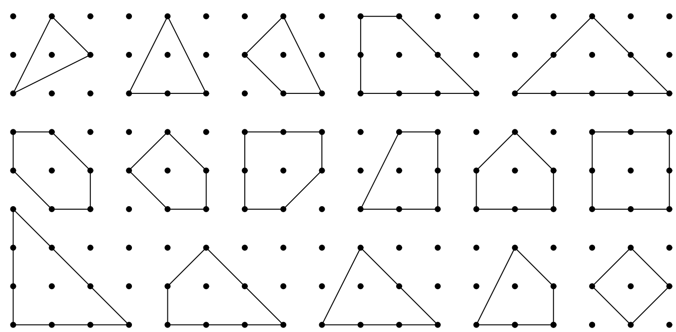
The objects of this problem are *reflexive polytopes*, which have a whole bunch of characterizations. In order to formulate these, we use the Ehrhart polynomial and its generating function $i_P(k) := \#(k\ P \cap \mathbb{Z}^d)$, $i_P^\circ(k) := \#(\mathrm{relint}(k\ P) \cap \mathbb{Z}^d)$, and $H_P(t) := \sum_{k \geq 0} i_P(k)\ t^k$. (See [**3**], and also Matthias Beck's section of this article). The Volume will always be normalized with respect to the lattice.

THEOREM 4.1. *Let $P \subset \mathbb{R}^d$ be a full–dimensional lattice polytope with unique interior lattice point* $\mathbf{0}$. *Then the following conditions are equivalent:*

(i) *The polar dual* $P^\vee = \{y \in (\mathbb{R}^d)^* \ : \ \langle y, P\rangle \geq -1\}$ *is integral.*
(ii) $\mathrm{vol}(P) = \sum \mathrm{vol}(F)$*, the sum ranging over all facets F of P.*
(iii) $i(P,k) = i^\circ(P, k+1)$ *for all k.*
(iv) $H(P, 1/t) = (-1)^{d+1} t H(P,t)$.
(v) *The projective toric variety X_P defined by P is Fano.*
(vi) *Every generic hypersurface of X_P is Calabi–Yau.*

The only non-elementary conditions (iv) and (v, vi) are due to Takeuchi Hibi [**4**] and Victor Batyrev [**1**] respectively. The polytope P is called *reflexive* if these conditions are satisfied. The last condition explaines the physicist's interest in reflexive polytopes.

In any given dimension, there is only a finite number of equivalence classes of reflexive polytopes. For example, these are the 16 reflexive polygons:



All reflexive polygons (up to $\mathrm{SL}_2\mathbb{Z}$).

Here is the first striking theorem.

THEOREM 4.2 (Cf. [**6**]). *The sum of the lengths of an admissible polygon and its dual is* 12.

In their beautiful paper [**6**], Bjorn Poonen and Rodriguez-Fernando Villegas give four proofs: exhaustion, walk in the space of reflexive polygons, toric surfaces, and modular forms. They also give an interpretation as a Gauß-Bonnet type theorem, where the curvature at a vertex is the length of the dual edge. Here comes the second striking result[1].

THEOREM 4.3. *If P is a* 3*-dimensional reflexive polytope, then*

$$\sum_{e \text{ edge of } P} \mathrm{length}(e) \cdot \ \mathrm{length}(e^\vee) = 24$$

"Proof": By the last characterization of reflexive polytopes, a generic hypersurface $Z \hookrightarrow X_P$ is a 2–dimensional Calabi–Yau, i.e., a $K3$ surface. Thus, $\chi(Z) = 24$. By [**2**], the above sum computes $\chi(Z)$. □

The only other proof known (to me) is the exhaustion proof. (There are 4,319 reflexive 3–polytopes [**5**].) I am not satisfied with this proof. There should be an

[1]Communicated by Dimitrios Dais

elementary proof out there. Again, one could interpret the length of the dual edge as a curvature, and it looks like a Gauß-Bonnet formula.

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Are smooth toric ideals quadratically generated?

Presented by Allen Knutson (UC Berkeley).

A lattice polytope $P \subset \mathbb{R}^d$ defines an ample line bundle L_P on a projective toric variety X_P. (See, e.g., [**4**, §3.4].) If X_P is smooth (the normal fan of P is unimodular), then L_P is very ample, and provides an embedding $X_P \hookrightarrow \mathbb{P}^{r-1}$, where $r = \#(P \cap \mathbb{Z}^d)$. So we can think of X_P canonically sitting in projective space. The following question about the defining equations of $X_P \subset \mathbb{P}^{r-1}$ has been around for quite a while, but its origins are hard to track.

Question 5.1. *Let P be a lattice polytope whose corresponding projective toric variety is smooth. Is the defining ideal I_P generated by quadratics?*

There are two variations of this question (of strictly increasing strength).

- Is the homogeneous coordinate ring $[X_1, \dots, X_r]/I_P$ Koszul?
- Does I_P have a quadratic Gröbner basis?

The last version has a combinatorial interpretation. It asks for the existence of very special, "quadratic" triangulations of P. These are regular unimodular triangulations whose minimal non-faces have two elements (they form a flag- or clique-complex) [**7**, Ch. 8].

Partial results were obtained by Robert Jan Koelman [**5**] who showed that Question 5.1 has an affirmative answer for polygons. Later, Winfried Bruns, Joseph Gubeladze and Ngô Viêt Trung [**2**] showed that smooth polygons even have quadratic triangulations. Lindsay Piechnik shows that smooth reflexive 4-polytopes have quadratic triangulations [**6**]. (See Christian Haase's section of this article for more about reflexive polytopes.)

Günter Ewald and Alexa Schmeinck [**3**] answer Question 5.1 affirmatively if the number of facets is at most $d+2$. They reformulate the problem as a pebble game, and provide a winning strategy. Given a set of n red and n green pebbles on the lattice points in P (possibly several pebbles at the same point) so that the barycenters of the red and the green set agree, is it possible to remove all the pebbles using the following two moves?

- If a red and a green pebble sit at the same lattice point, remove them from the game.
- Two red (or green) pebbles can move in opposite directions (so as to preserve the barycenter), provided they stay inside of P.

This is actually the first reference we could find where Question 5.1 was asked explicitly.

Rikard Bögvad announced (and later withdrew) a general proof of 5.1 [**1**].

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The classification of 1-point lattice tetrahedra.

Presented by Bruce Reznick (University of Illinois at Urbana-Champaign).

A **k-point lattice n-simplex** is a simplex $P = conv(v_0, \dots, v_n)$ with $v_j \in \mathbb{Z}^n$, and so that $P \cap \mathbb{Z}^n$ consists of the vertices and exactly k interior lattice points. For $n = 2$, Pick's Theorem says that a k-point lattice triangle has area $k + \frac{1}{2}$; conversely, any plane triangle with that area, lattice point vertices and no other lattice points on its edges is a k-point lattice triangle. Thus, the first interesting case is $n = 3$. We restrict our attention mostly to $k = 0$ ("empty lattice tetrahedra") and to $k = 1$.

For $(a, b, c) \in \mathbb{Z}^3$, let

$$T_{a,b,c} := conv((0, 0, 0), (1, 0, 0), (0, 1, 0), (a, b, c)).$$

In 1957, J. Reeve [**9**] noted that for $n \in \mathbb{N}$, $T_{1,1,n}$ must be empty, and since it has volume $n/6$, there is no upper bound on the volume of empty lattice tetrahedra. There are several equivalent, but somewhat different, characterizations of empty tetrahedra, due to G. White, R. Howe and H. Scarf (see the discussion in [**10**]), to independently to D. Handelman [**1**] and D. Morrison and G. Stevens [**7**]. One way of saying it is that the tetrahedron $T = conv(v_0, v_1, v_2, v_3)$ is empty if and only if there is a unimodular affine transformation taking T to $T_{0,0,1}$ or to $T_{1,b,c}$, with $1 \le b < c$ and $gcd(b, c) = 1$.

In 1983, D. Hensley [**2**] proved that for each fixed $k \ge 1$ and n, there is an upper bound on the volume of k-point lattice n-simplices. His bound was improved in 1990 by J. Lagarias and G. Ziegler [**4**]. Lagarias suggested [**3**] at Snowbird that this counterintuitive phenomenon regarding the volume of empty and non-empty simplices is worthy of new attention.

If P is an n-simplex, then the set $\{v_j - v_0 : 1 \le j \le n\}$ spans $\mathbb{R}^n$, hence for any $w \in \mathbb{R}^n$, there exist unique $\lambda_j \in \mathbb{R}$ so that $w - v_0 = \sum_{j=1}^n \lambda_j(v_j - v_0)$. Putting $\lambda_0 := 1 - \sum_{j=1}^n \lambda_j$, we see that each $w \in \mathbb{R}^n$ has a unique representation $w = \sum_{j=0}^n \lambda_j v_j$ with $\sum_{j=0}^n \lambda_j = 1$. The λ_j's are called the **barycentric coordinates** of w with respect to the vertices of P. Note that $w \in P$ if and only if $\lambda_j \ge 0$ for $j = 0, \dots, n$ and $w \in int(P)$ if and only if $\lambda_j > 0$.

Suppose P is a k-point lattice n-simplex. The barycentric coordinates of any interior lattice point are necessarily positive rational numbers. The author proved in [**10**] that there is an upper bound to the denominator of the barycentric coordinates, and so for fixed (k, n), only finitely many different sets of barycentric coordinates are possible. It was also shown in [**10**] that, up to permutation, there are exactly 7 possible barycentric coordinates for the interior point of a 1-point lattice tetrahedron, namely:

$$\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right), \left(\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\right), \left(\tfrac{3}{7}, \tfrac{2}{7}, \tfrac{1}{7}, \tfrac{1}{7}\right), \left(\tfrac{5}{11}, \tfrac{3}{11}, \tfrac{2}{11}, \tfrac{1}{11}\right),$$

$$\left(\tfrac{5}{13}, \tfrac{4}{13}, \tfrac{3}{13}, \tfrac{1}{13}\right), \left(\tfrac{7}{17}, \tfrac{5}{17}, \tfrac{3}{17}, \tfrac{2}{17}\right), \left(\tfrac{7}{19}, \tfrac{5}{19}, \tfrac{4}{19}, \tfrac{3}{19}\right).$$

A natural (and open!) question is to characterize, up to unimodular transformation, the 1-point lattice tetrahedra. This would also determine the maximal possible volume. The estimate from [**4**, p.1023] gives the maximum volume as $\le 14^{48} \approx 10^{55}$. This has been very recently improved by O. Pikhurko [**8**, p.23] to $< \frac{85}{6}$. It is noted in [**10**, p.235] that $T_{-13,-9,20}$ is a 1-point lattice tetrahedron, so the volume can be as large as $\frac{20}{6}$. This question was also discussed in a non-explicit way by J. Lawrence [**5**].

This question appears to be large, but tractable. Let T have vertices $\{v_j\}$ and interior point w. Then $T' = conv(v_0, v_1, v_2, w)$ is necessarily an empty tetrahedron, and so can be identified via White-Scarf-Howe-Morrison-Stevens. Since w has one of the barycentric coordinates listed above, we can then solve for v_3 and check whether T contains any unwanted lattice points.

Here is the simplest case. Suppose $T' = T_{1,1,n}$, and suppose that $w = (1, 1, n) = \frac{1}{4}(v_0+v_1+v_2+v_3)$ is the centroid of T. Then $v_3 = 4w-(v_0+v_1+v_2)$ and $T = T_{3,3,4n}$. Since T lies between the planes $x_1 - x_2 = \pm 1$ with a single vertex on each, any non-vertices (a, b, c) in $T \cap \mathbb{Z}^3$ must have $a = b = 1$ or $a = b = 2$. An easy computation shows that

$$(1, 1, c) = \tfrac{5c-4n}{4n}(0, 0, 0) + \tfrac{4n-3c}{4n}(1, 0, 0) + \tfrac{4n-3c}{4n}(0, 1, 0) + \tfrac{c}{4n}(3, 3, 4n),$$

hence $(1, 1, c)$ is in $T \cap \mathbb{Z}^3$ provided $\frac{4}{3}n \ge c \ge \frac{4}{5}n$. When $n \le 2$, this gives only $c = n$, but for $n \ge 3$, $c = n+1$ satisfies the inequality, hence $(1, 1, n+1)$ is in T and T is not a 1-point lattice tetrahedron. Similarly,

$$(2, 2, c) = \tfrac{5c-12n}{4n}(0, 0, 0) + \tfrac{8n-3c}{4n}(1, 0, 0) + \tfrac{8n-3c}{4n}(0, 1, 0) + \tfrac{c}{4n}(3, 3, 4n)$$

is in $T \cap \mathbb{Z}^3$ provided $\frac{8}{3}n \ge c \ge \frac{12}{5}n$. For $n = 1$, there is no such point, but for $n = 2$, $c = 5$ satisfies the inequality. Thus, in this simplest case, the only 1-point lattice tetrahedron is $T_{3,3,4}$, which is unimodular, under $(x_1, x_2, x_3) \mapsto (x_1-x_3, x_2-x_3, x_3)$ to $T_{-1,-1,4}$. This tetrahedron also appears in [**10**, p.235].

More has already been done in this direction. As a 2002 undergraduate research project at the University of Michigan under the direction of S. Bullock, B. Mazur [**6**] analyzed the centroid case in complete detail. He showed that if T is a 1-point lattice tetrahedron whose interior point is the centroid, then it is unimodularly equivalent either to $T_{3,3,4}$ or to $T_{7,11,20}$.

Note added in proof. The characterization of 1-point lattice tetrahedra has been completed by Alexander M. Kasprzyk in the paper "Toric Fano 3-folds with terminal singularities", `arXiv.math.AG/0311284`, 17 Nov 2003. Each of the 6 non-centroid sets of barycentric coordinates has, up to unimodular equivalence, a unique representative 1-point lattice tetrahedron. Thus, there are 8 different examples in all. The author is grateful to Julian Pfeifle for this reference. The author had independently reached the same conclusion, in a forthcoming paper to be called "Clean lattice tetrahedra".

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A conjecture on lattice tiles.

Presented by Sinai Robins (Temple University).

We first define a tiling of the n-dimensional integer lattice $\mathbb{Z}^n$ by "lattice tiles" and then formulate a conjecture about the structure of the tiles for $n = 2$.

We begin by writing the lattice $\mathbb{Z}^n$ as a finite, disjoint union of subsets:

$$\mathbb{Z}^n = \bigcup_{j=1}^{N} \{L_j + v_j\}$$

for some N, some collection of sublattices $L_j \subset \mathbb{Z}^n$, and some collection of integer translation vectors v_j. Each of these subsets $L_j + v_j$ is called a **lattice tile**.

For $n = 1$, there have been several papers [**1, 2, 3**] on tiling (or covering) $\mathbb{Z}$ by such lattice tiles, which are of course arithmetic progressions in the 1-dimensional case. Here we extend the notion of covers of the integers to tilings of the lattice $\mathbb{Z}^n$.

A recent paper of Zhi-Wei Sun [**4**] proves, using group-theoretic considerations, that if a group is given as the disjoint union of various translates of subnormal subgroups, then at least two of those subgroups must have the same index. Applying this theorem to the group $\mathbb{Z}^n$ given by a lattice tiling, we immediately see that at least two of the lattice tiles must have the same index. In other words, we know that at least two of the lattice tiles must have the same volume for their fundamental domain. We conjecture that for $n = 2$ more is true:

CONJECTURE 7.1. *Given any lattice tiling of $\mathbb{Z}^2$, at least two of the tiles must be translates of each other.*

That is, while the result of Sun [**4**] guarantees the existence of two tiles with the same area for their fundamental domain, experimentation suggests a refined statement about the shape of the corresponding fundamental domains.

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Covering minima, free planes and deep holes

Presented by ACHILL SCHÜRMANN (Magdeburg).

In [**3**] Kannan and Lovász introduced the *covering minima*

$$\mu_i(K, \Lambda) = \min\{\mu > 0 \ : \ \Lambda + \mu K \text{ meets every } (n-i)\text{–dim. aff. subspace in } \mathbb{R}^n\},$$

$i = 1, \ldots, n$, of a (full rank) lattice Λ and a convex body K in $\mathbb{R}^n$, generalizing the previously known *covering radius* $\mu_n(K, \Lambda)$.

A problem, arising for example in the study of *free lattice planes* (cf. [**1**], [**2**], [**4**]), is to compute $\mu_i(K, \Lambda)$. A free lattice plane with respect to Λ and K is an affine subspace in $\mathbb{R}^n \setminus (\Lambda + K)$. Thus a free plane of dimension $(n-i)$ exists if and only if $\mu_i(K, \Lambda) > 1$. Of particular interest is the case where $K = B^n$ is the (solid) unit sphere.

It is known (cf. [**3**]) that

$$\mu_i(K, \Lambda) = \max\{\, \mu_i(K|L^\perp, \Lambda|L^\perp) \ : \ L \text{ is a } (n-i)\text{–dim. sublattice of } \Lambda \,\},$$

where $\cdot|L^\perp$ denotes the orthogonal projection onto the linear subspace $L^\perp$, orthogonal to L. Therefore, it is possible to show that $\mu_i(K, \Lambda)$ can theoretically be obtained by computing finitely many covering radii in dimension i. This approach is unpractical though.

Maybe another interpretation helps here. If $K = B^n$, then the covering radius $\mu_n(B^n, \Lambda)$ has two nice geometrical meanings: On the one hand, the covering radius is equal to the maximum (Euclidean) norm $\|v\|$ among the vertices v of the *Dirichlet–Voronoi–polytope*

$$\{\, x \in \mathbb{R}^n \ : \ \|x\| \le \|x - y\| \text{ for all } y \in \Lambda \,\}.$$

These vertices are called the *deep holes* of Λ. On the other hand, the covering radius is equal to the circumradius of some *Delone–polytope* of Λ — a Λ–polytope with its circumball containing its vertices on the boundary and beyond that no other lattice points (see Section "Lattice polytopes: some open problems" by Jean–Michel Kantor).

Both, the Dirichlet–Voronoi–polytope and the finitely many Delone–polytopes yield classical tilings of $\mathbb{R}^n$ — dual to each other — by lattice point translations. Thus an affine subspace attaining a covering minimum runs necessarily through a Dirichlet–Voronoi–polytope and through some Delone–polytopes. This gives reason to propose

PROBLEM 8.1. *Find a geometrical description of affine subspaces attaining $\mu_i(B^n, \Lambda)$, depending on Λ's Dirichlet–Voronoi–polytope or its Delone–polytopes.*

In the planar case for example, lines attaining $\mu_1(B^2, \Lambda)$ run through midpoints of neighboring facets (bounding segments) of the Dirichlet–Voronoi–polytope or contain a facet. Depending on such a geometrical description or not, it would be nice to solve

PROBLEM 8.2. *Find a way to compute the covering minima of a lattice, e.g. of the root lattice E_8 or the Leech lattice.*

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