Lattice polygons, the number 12, and onion skins.

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Joint with Josef Schicho (RISC Linz)
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Pick's formula

Examples

12

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1. Pick’s formula

**Proposition 1** Every triangle without lattice points but the vertices has area \( a = \frac{1}{2} \left( = i + \frac{b}{2} - 1 \right) \).
‘Proof’: 

area of triangle \( \begin{pmatrix} 0 & p & r \\ 0 & q & s \end{pmatrix} \) 

\[ = \frac{1}{2} \text{ area of parallelogram } \begin{pmatrix} 0 & p & r+r \\ 0 & q & s+s \end{pmatrix} \]

\[ = \frac{1}{2} \left| \det \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right|. \]
The translates of the parallelogram by \((\frac{p}{q})\) and \((\frac{r}{s})\) tile the plane.

Because there are no lattice points in the parallelogram (other than the vertices), \((\frac{p}{q} \frac{r}{s})\) has integral inverse.  

All empty triangles are equivalent by lattice preserving affine maps \(SL_2\mathbb{Z} \times \mathbb{Z}^2\). \((x \mapsto Ax + c, A \in SL_2\mathbb{Z}, c \in \mathbb{Z}^2.)\)
Theorem 2 (Pick 1899)

\[ a = i + \frac{b}{2} - 1 \]
Caveat!
2. Examples

\[ i = 0: \]

\[ i = 1: \quad b \leq 9 = 2i + 7 \]

\[ i \geq 2: \quad b \leq 2i + 6 \]

Theorem 3 (Scott 1967, Schicho 2000, 2001) This classifies triples \((a, i, b)\) that come from convex lattice polygons.
3. “12”


**Definition 4** A primitive segment \( s = (x, y) \), is admissible if the triangle \((0, x, y)\) contains no other lattice points. The length of \( s \) is the determinant \( \det[x, y] = \pm 1 \).

\[
x + 2y = 1
\]

\[
\begin{vmatrix}
  -1 & 1 \\
  1 & 0
\end{vmatrix} = -1
\]

some non-admissable segments the dual of an admissible segment
Theorem 5 (Poonen and Villegas 2000) The sum of the lengths of an admissible polygon and its dual is $12$ (times the winding number).
All reflexive polygons (up to $SL_2\mathbb{Z}$).
4. Onion skins

Levels $\ell = 3$, $\ell = 2$, $\ell = 5/2$, $\ell = 7/3$, and $\ell = 8/3$.

**Theorem 6 (2003)**

$$ (2\ell - 1)b \leq 2i + 9\ell^2 - 2 $$

with equality if and only if $P$ is a multiple of a unimodular triangle.
Lemma 7

\[ b \leq b^{(1)} + 9 \]

(With equality only for multiples of a unimodular triangle.)

‘Proof’:

[Diagram showing a grid with arrows and points, illustrating the proof of the lemma.]
‘Proof’ of Theorem:

\[(2\ell - 1)b \leq (2\ell - 1)b^{(1)} + 9(2\ell - 1)\]
\[= 2b^{(1)} + (2(\ell - 1) - 1)b^{(1)} + 9(2\ell - 1)\]
\[\leq 2b^{(1)} + 2i^{(1)} + 9(\ell - 1)^2 - 2 + 9(2\ell - 1)\]
\[= 2i + 9\ell^2 - 2\]
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