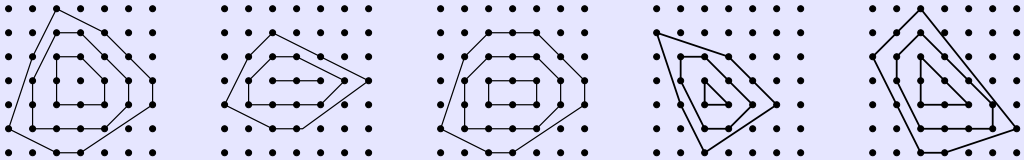
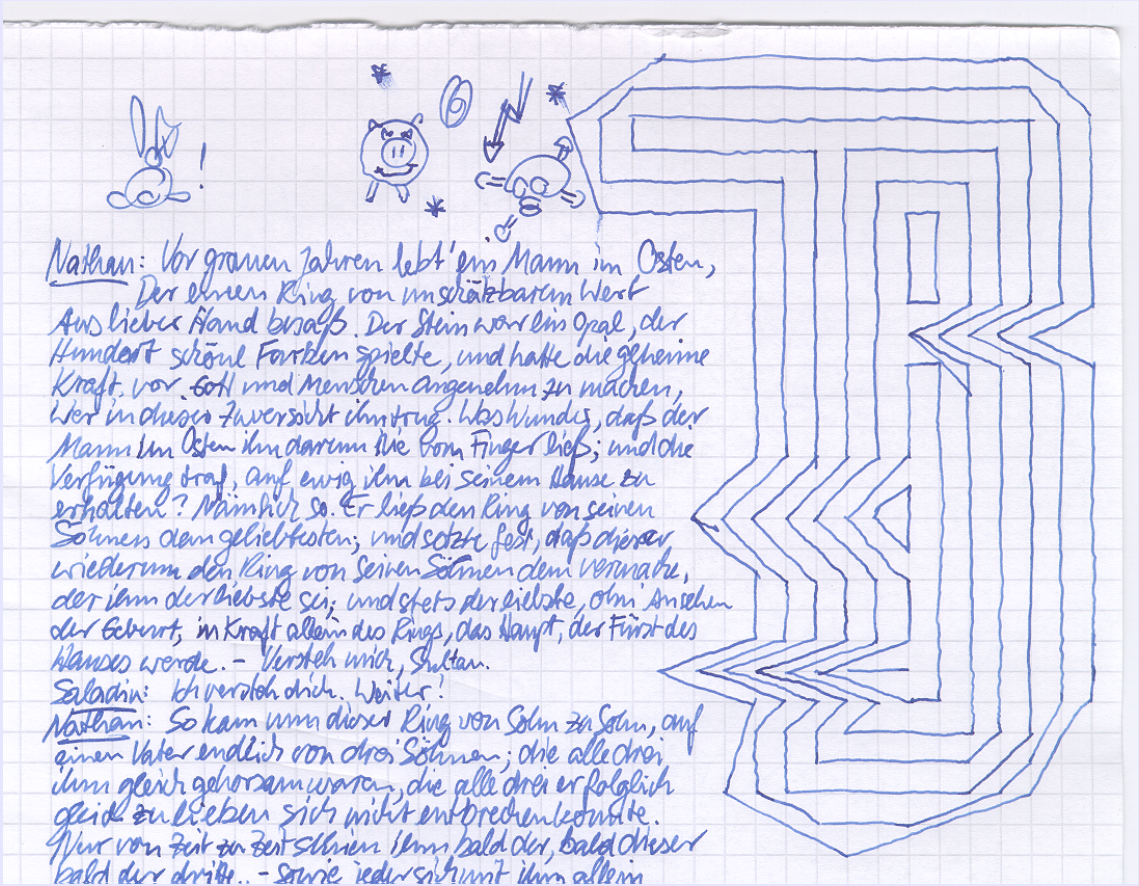
Lattice polygons, the number 12, and onion skins.

Christian Haase

October 3, 2003



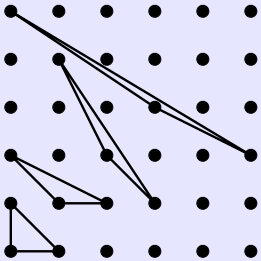
Joint with Josef Schicho (RISC Linz)



Nathan: Vor grauen Jahren lebt' ein Mann im Osten,
Der einen Ring von unschätzbarem Wert
Aus lieber Hand besaß. Der Stein war ein Opal, der
Hundert schöne Farben spielte, und hatte die geheime
Kraft, vor Gott und Menschen angenehm zu machen,
Wer in diesem Zuversicht ihn trug. Was Wunder, daß der
Mann im Osten ihn darum nie vom Finger hieß; und die
Verfügung traf, auf ewig ihn bei seinem Hause zu
erhalten? Nämlich so. Er ließ den Ring von seinen
Söhnen dem geliebtesten; und setzte fest, daß dieser
wiederum den Ring von seinen Söhnen dem vermache,
der ihm der liebste sei; und stets der geliebte, ohn' Ansehen
der Geburt, in Kraft allein des Rings, das Haupt, der Fürst des
Hauses werde. – Versteh mich, Sultan.
Saladin: Ich verstehe dich. Weiter!
Nathan: So kam nun dieser Ring von Sohn zu Sohn, auf
einen Vater endlich von drei Söhnen; die alle drei
ihm gleich gehorsam waren, die alle drei er folglich
gleich zu lieben sich nicht entbrechen konnte.
Nur von Zeit zu Zeit schien ihm bald der, bald dieser
bald der dritte.. – sowie jeder sich mit ihm allein

1. Pick's fomula

Proposition 1 *Every triangle without lattice points but the vertices has area* $a = 1/2 \left(= i + b/2 - 1 \right)$.

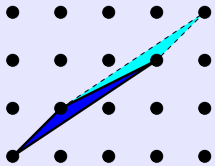


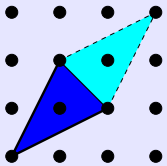
'Proof':

area of triangle $\begin{pmatrix} 0 & p & r \\ 0 & q & s \end{pmatrix}$

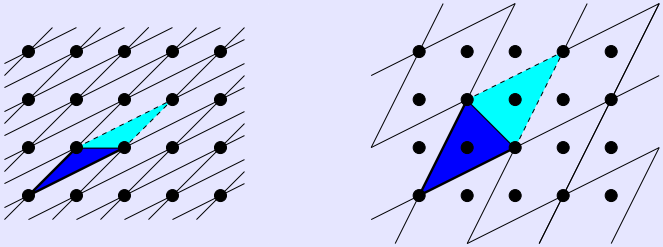
$$= \frac{1}{2} \text{ area of parallelogram } \begin{pmatrix} 0 & p & r & p+r \\ 0 & q & s & q+s \end{pmatrix}$$

$$= \frac{1}{2} \left| \det \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right|.$$





The translates of the parallelogram by $\binom{p}{q}$ and $\binom{r}{s}$ tile the plane.

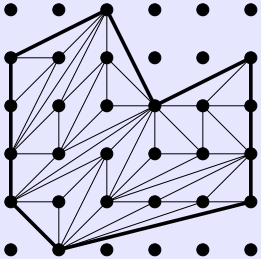


Because there are no lattice points in the parallelogram (other than the vertices), $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$ has integral inverse. $\square$

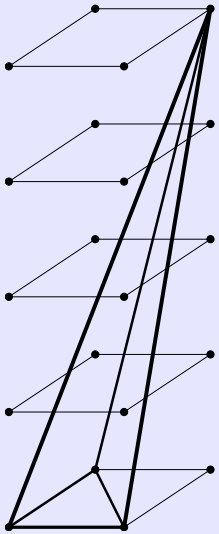
All empty triangles are equivalent by lattice preserving affine maps $\mathrm{SL}_2\mathbb{Z} \ltimes \mathbb{Z}^2$. ($x \mapsto Ax + c$, $A \in \mathrm{SL}_2\mathbb{Z}$, $c \in \mathbb{Z}^2$.)

Theorem 2 (Pick 1899)

$$a = i + \frac{b}{2} - 1$$

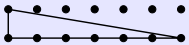


Caveat!

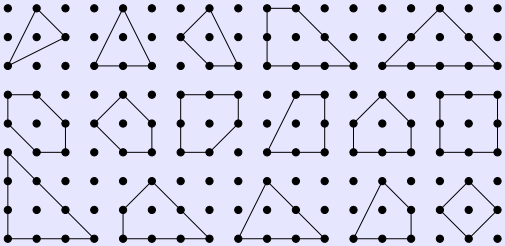


2. Examples

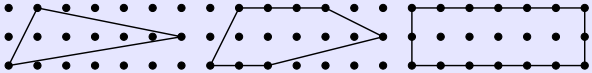
$i = 0$:



$i = 1$: $b \leq 9 = 2i + 7$



$i \geq 2$: $b \leq 2i + 6$



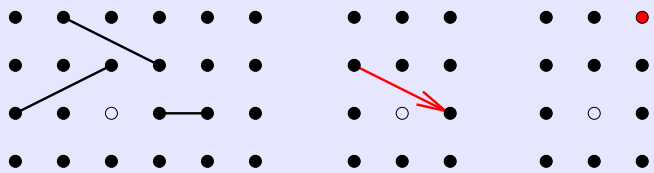
Theorem 3 (Scott 1967, Schicho 2000, 2001) *This classifies triples* (a, i, b) *that come from convex lattice polygons.*

3. "12"

[Poonen, Villegas: *Lattice polygons and the number 12*; Amer. Math. Monthly 2000]

Definition 4 A primitive segment $s = (\mathbf{x}, \mathbf{y})$, is admissible if the triangle $(\mathbf{0}, \mathbf{x}, \mathbf{y})$ contains no other lattice points.
The length of s is the determinant $\det[\mathbf{x}, \mathbf{y}] = \pm 1$.

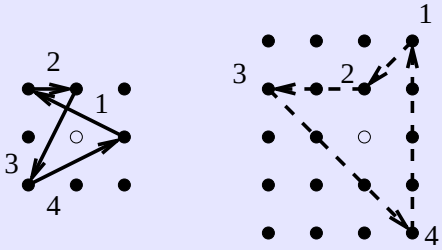
$$x + 2y = 1$$

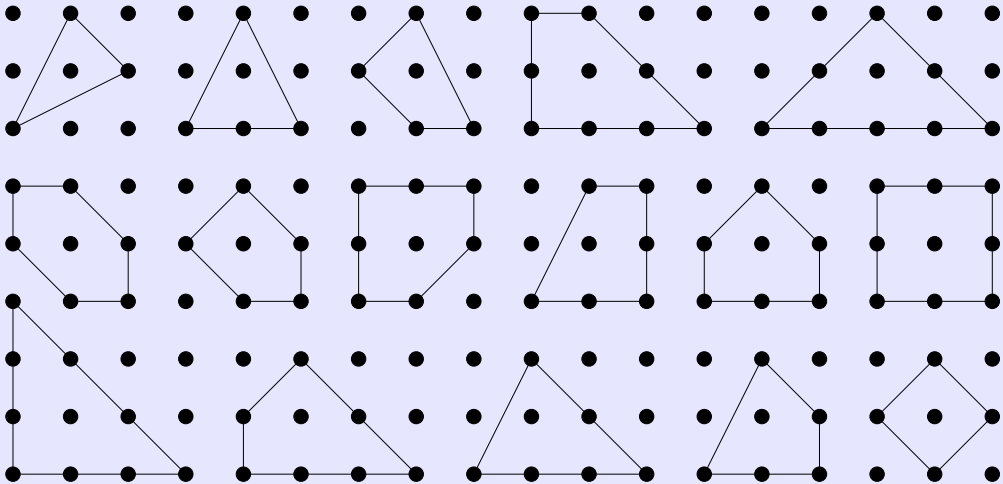


$$\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

some non-admissable segments the dual of an admissible segment

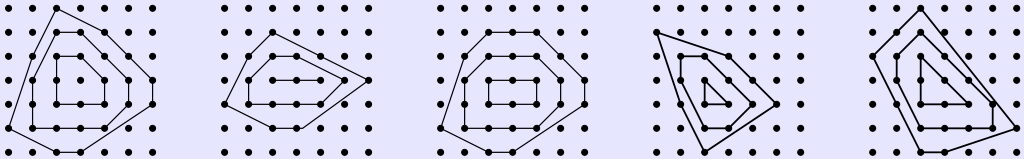
Theorem 5 (Poonen and Villegas 2000) *The sum of the lengths of an admissible polygon and its dual is* 12 *(times the winding number).*





All reflexive polygons (up to $\mathrm{SL}_2\mathbb{Z}$).

4. Onion skins



Levels $\ell = 3$, $\ell = 2$, $\ell = 5/2$, $\ell = 7/3$, and $\ell = 8/3$.

Theorem 6 (🐰 2003)

$$(2\ell - 1)b \leq 2i + 9\ell^2 - 2$$

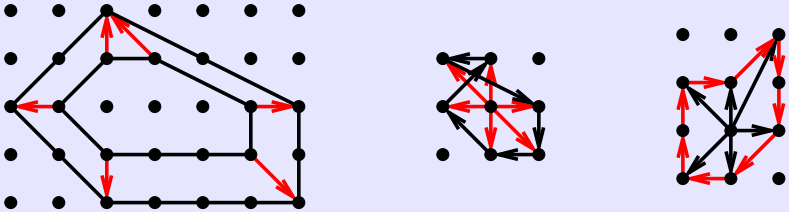
with equality if and only if P is a multiple of a unimodular triangle.

Lemma 7

$$b \leq b^{(1)} + 9$$

(With equality only for multiples of a unimodular triangle.)

'Proof':



$\square$

'Proof' of Theorem:

$$
\begin{aligned}
(2\ell-1)b &\leq (2\ell-1)b^{(1)} + 9(2\ell-1) \\
&= 2b^{(1)} + (2(\ell-1)-1)b^{(1)} + 9(2\ell-1) \\
&\leq 2b^{(1)} + 2i^{(1)} + 9(\ell-1)^2 - 2 + 9(2\ell-1) \\
&= 2i + 9\ell^2 - 2
\end{aligned}
$$

$\square$