Polar decomposition and Brion's theorem.

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ABSTRACT. In this note we point out the relation between Brion's formula for the lattice point generating function of a convex polytope in terms of the vertex cones [**Bri88**] on the one hand, and the polar decomposition à la Lawrence/Varchenko [**Law91a, Var87**] on the other. We then go on to prove a version of polar decomposition for non-simple polytopes.

1. Introduction

The stars of this note are two formulas that express convex polytopes in terms of cones. On the one hand, Brion's theorem (equation (2) below) expresses lattice points in a polytope in terms of tangent cones at vertices [**Bri88**]. On the other hand, given a simple polytope and a generic objective function, one can write the indicator function of a simple polytope as a combination of polarized tangent cones at the vertices (equation (4)). We refer to this formula as polar decomposition [**Law91a, Var87**].

After we review these formulas in this introduction, we point out in section 2 some simple relations between the two formulas which, to our knowledge, have not appeared in the literature before. In particular, Brion's theorem (for simple polytopes) follows from polar decomposition. The last section is devoted to the formulation and proof of polar decomposition for non-simple polytopes[1].

This is by no means an attempt to survey what is scattered throughout the literature[2] about decompositions of polytopes into cones. In

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[1]Put together, sections 2 and 3 yield another proof of Brion's theorem.

[2]See [**Bar02**, p. 346] and the references therein for Brion's theorem, [**Law91a, Var87**] for polar decomposition, and [**Bri37, Gra74, She67**] for the Brianchon-Gram formula below.

fact, Matthias Beck made me aware of such a survey in the making [**BRSW04**]. We will concentrate on what we think is new. We will not state the most general form of the results, but leave generalizations (e.g., weighted versions) as exercises for the ambitious reader.

Acknowledgments. This paper has its origins in José Agapito's inspiring talk about his weighted version of polar decomposition for simple polytopes [**Aga03**]. After the talk, I asked the question whether one can obtain an analogous formula for non-simple polytopes by considering simple deformations. I got intrigued by this question, and, after computing several examples, was convinced that there is such a theorem out there.

I was lucky to get hold of a preliminary draft of the above mentioned survey [**BRSW04**]. Not only did it take away the pressure to try and write a comprehensive article about conic decompositions of polytopes, but it also made me aware of references and view points that I did not know about before. Much of the presentation in the present paper is influenced by (if not stolen from) this draft. Finally, I want to thank the referee who helped to clarify the exposition.

1.1. Basic definitions and the Brianchon-Gram formula.
We will consider rational convex polytopes P and polyhedral cones C in $\mathbb{Q}^d$, where $\mathbb{Q}^d$ is endowed with the fixed lattice $\mathbb{Z}^d$. We will assume that polytopes and cones are full-dimensional. For standard polytope definitions and notation we refer to [**Zie95**]. To a subset $S \subseteq \mathbb{Q}^d$ we assign two objects:

- The indicator function $\mathbb{1}_S \colon \mathbb{Q}^d \to \mathbb{Z}$ vanishes outside S, and is one along S.
- The generating function (of the lattice points) is the formal Laurent power series $G_S = \sum_{m \in S \cap \mathbb{Z}^d} z^m \in \mathbb{C}[[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]]$, where we write $z^m = z_1^{m_1} \cdot \ldots \cdot z_d^{m_d}$. If $S = P$ is a polytope, then G_P is a Laurent polynomial; when $S = C$ is a cone, then G_C is a rational function[3].

The essential features can be illustrated on the line. For $a < b \in \mathbb{Z}$, we have $G_{[a,b]}(x) = x^a + \ldots + x^b = \frac{x^a - x^{b+1}}{1-x}$, and $G_{[a,\infty)}(x) = x^a + \ldots = \frac{x^a}{1-x}$.

The mother of all conic decomposition theorems is the Brianchon-Gram formula. According to Shephard [**She67**], Brianchon [**Bri37**] and Gram [**Gra74**] independently proved the $d = 3$ case in 1837(!) and 1874 respectively. In 1927 Sommerville [**Som27**] published a proof for

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[3]We are glossing over significant technicalities here and later on when we pass from formal power series to rational functions. For a first reading, we prefer to skip this step, and refer to [**Bar02, BV97**] for a careful treatment. The experts should be familiar with the standard arguments, anyway.

general d, which was corrected in the 1960's by Grünbaum [**Grü03**, §14.1].

Let $P \subset \mathbb{Q}^d$ be a convex polytope. For a face F of P, we define the tangent cone (of P at F) by

$$\mathcal{T}_F P = \{ f + x \in \mathbb{Q}^d \ : \ f \in F \text{ and } f + \varepsilon x \in P \text{ for some } \varepsilon > 0 \}$$

(See Figure 1.) Then $\mathbb{1}_P$ is the alternating sum of all tangent cones:
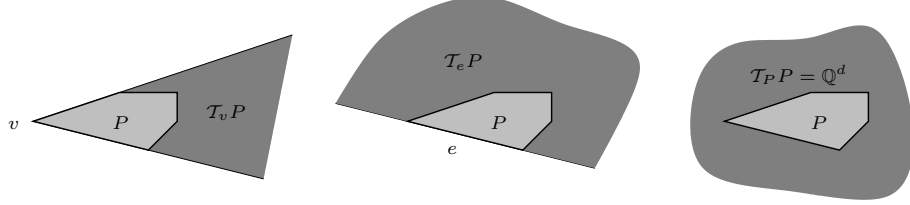

FIGURE 1. Tangent cones at vertex v, edge e, and at P.

$$(1) \qquad \mathbb{1}_P = \sum_{F \preceq P} (-1)^{\dim F} \, \mathbb{1}_{\mathcal{T}_F P}.$$

At every point $x \in P$, the right hand side computes the Euler characteristic of P (and P is contractible). While outside P, we have to subtract the Euler characteristic of the subcomplex that is visible from x (which again, is contractible, actually shellable). (Compare [**She67**].)

1.2. Brion's formula. Brion [**Bri88**] proved his decomposition using a Riemann-Roch type formula. In the mean time, more elementary proofs have been given [**Bar93, BV97, Law91b, PK92**]. The easiest way to formulate Brion's formula is in terms of generating functions. We can express the generating function for P as the sum of the generating functions of the tangent cones at vertices.

$$(2) \qquad G_P = \sum_{v \text{ vertex of } P} G_{\mathcal{T}_v P}$$

In the example, $\mathcal{T}_a[a, b] = [a, \infty)$, and $\mathcal{T}_b[a, b] = (-\infty, b]$ with generating functions $G_{[a,\infty)} = \frac{x^a}{1-x}$ and $G_{(-\infty,b]} = \frac{(1/x)^{-b}}{1-1/x} = -\frac{x^{b+1}}{1-x}$. As predicted, they sum to $G_{[a,b]}$.

This version of Brion's formula is implied by the following two results. (Compare [**Bar02**].)

THEOREM 1. *The function $\mathbb{1}_P - \sum_{v \text{ vertex of } P} \mathbb{1}_{\mathcal{T}_v P}$ is a linear combination of indicator functions of cones that contain affine lines.*

This follows immediately from (1). It is also implied by the considerations in Sections 2 and 3. In our example, $\mathbb{1}_{[a,b]} = \mathbb{1}_{[a,\infty)} + \mathbb{1}_{(\infty,b]} - \mathbb{1}_{\mathbb{Q}}$.

LEMMA 2. *Let $C \subseteq \mathbb{Q}^d$ be a cone that contains an affine line. Then $G_C = 0$.*

The idea of the proof is to decompose the monoid $C \cap \mathbb{Z}^d$ into a direct sum $L \oplus R$ of a one dimensional lattice L and a complement R. Then G_C can be written as $G_R G_L$, and $G_L = 0^4$.

1.3. (Simple) polar decomposition. By flipping the edge vectors emanating from each vertex of P in a systematic way, we can get the so-called polar decomposition, which expresses the characteristic function of a convex simple polytope in terms of the characteristic functions of the cones supported at the vertices of P (no cones with straight lines needed this time). This was first obtained by Varchenko [**Var87**] and Lawrence [**Law91a**] independently. Lawrence used Brianchon-Gram's theorem together with the principle of inclusion-exclusion to derive the polar decomposition theorem. Then Karshon, Sternberg and Weitsman obtained a weighted version of this decomposition by assigning, in a consistent way, particular weights to the lattice points in the polytope and in the cones [**KSW03**]. Their work was also motivated by methods in differential and algebraic geometry. Finally, using the same source of motivation, Agapito [**Aga03**] gave a more general weighted version of the polar decomposition theorem for simple polytopes that includes Lawrence/Varchenko and Karshon-Sternberg-Weitsman versions as particular cases.

Polar decomposition is like Morse theory for simple polytopes. We sweep over the polytope and build it up from local contributions, critical point (vertex) by critical point. As input we use a generic[5] linear functional $\xi \in (\mathbb{Q}^d)^*$. Using ξ, we define the polarized tangent cones as follows.

Because P is simple, at every vertex v, the tangent cone $\mathcal{T}_v P$ is generated by d linearly independent directions $t_1, \ldots, t_d$ (remember P is full-dimensional).

$$\mathcal{T}_v P = v + \sum_{i=1}^{d} \mathbb{Q}_{\geq 0} t_i$$

Now the polarized tangent cone (with respect to ξ) is

$$\mathcal{T}_v^\xi P = v + \sum_{\xi(t_i)>0} \mathbb{Q}_{\geq 0} t_i + \sum_{\xi(t_i)<0} \mathbb{Q}_{<0} t_i.$$

This is a locally closed cone all whose points are ξ-higher than v.

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[4]Glossing over alert: look under the rug [**Bar02**]!

[5]Generic means here that ξ is not constant on any edge of P. In Section 3 we ask a little bit more.
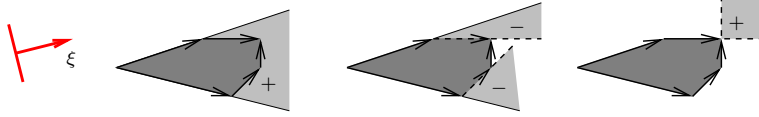
FIGURE 2. Polarized tangent cones.

(Compare Figure 2.) Dually, if $n_1, \ldots, n_d \in (\mathbb{Q}^d)^*$ are the inner facet normals to P at v, and $\xi = \sum \alpha_i n_i$, then $\mathcal{T}_v^\xi P$ can be defined by the inequalities

(3) $n_i(x) \geq n_i(v)$ for $\alpha_i > 0$, and $n_i(x) < n_i(v)$ for $\alpha_i < 0$.

Define the index $\mathrm{ind}_\xi(v)$ of the vertex v as the number of ξ-negative edge directions t_i, or equivalently, as the number of negative coefficients α_i. Then we can write P as the signed sum of polarized tangent cones.

(4) $$\mathbb{1}_P = \sum_{v \text{ vertex of } P} (-1)^{\mathrm{ind}_\xi(v)} \mathbb{1}_{\mathcal{T}_v^\xi P}$$

Again, we can formulate a generating function version of this result: just replace all $\mathbb{1}$'s by G's[6].

    Equation (4) follows from (1) if we group all faces according to where they achieve their ξ-maximum. Then it remains to check that

(5) $$(-1)^{\mathrm{ind}_\xi(v)} \mathbb{1}_{\mathcal{T}_v^\xi P} = \sum_{\substack{v \preceq F \preceq P \\ \xi(F) \leq \xi(v)}} (-1)^{\dim F} \, \mathbb{1}_{\mathcal{T}_F P}.$$

There is a weighted version of equation (4) [**Aga03**]. The weighted indicator function of a polytope P is the function $\mathbb{1}_P^w \colon \mathbb{Q}^d \to \mathbb{Z}[z]$ which vanishes outside P, and takes the value z^k along the relative interior of codimension k faces.[7] One could even introduce a different variable for every facet of P, and assign their product to the corresponding intersection. Now the same formula holds if we modify the indicator functions of the polarized tangent cones as follows: A face of (the closure of) $\mathcal{T}_v^\xi P$ is defined as the set of points that satisfy equality for some of the inequalities (3). The value of $\mathbb{1}_{\mathcal{T}_v^\xi P}^w$ along a face defined by k_+ equalities $n_i(x) = n_i(v)$ for $\alpha_i > 0$ and k_- equalities for $\alpha_i < 0$ is the polynomial $z^{k_+}(1 - z)^{k_-} \in \mathbb{Z}[z]$. One recovers the unweighted version for $z = 1$, and for $z = 0$ one obtains a decomposition of the interior of P.

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[6]More glossing over.

[7]The formula in [**Aga03**] is stated using the substitution $z = \frac{1}{1+y}$.

2. This follows from that

In this section we show the relation between the two formulas. Essentially, for the indicator functions, Brion's formula (in the simple case) is polar decomposition modulo cones that contain lines.

LEMMA 3. *If P is a simple polytope and v is a vertex, then the different polarized tangent cones for various ξ partition $\mathbb{Q}^d$. Moreover, for $\xi_1, \xi_2 \in (\mathbb{Q}^d)^*$, $(-1)^{\mathrm{ind}_{\xi_1}(v)} \mathbb{1}_{\mathcal{T}_v^{\xi_1} P}$ and $(-1)^{\mathrm{ind}_{\xi_2}(v)} \mathbb{1}_{\mathcal{T}_v^{\xi_2} P}$ are equivalent modulo cones that contain lines.*

PROOF. If $n_i(x) \geq n_i(v)$ for $i = 1, \ldots, d$ are the inequalities of the facets incident to v, the hyperplanes $n_i(x) = n_i(v)$ subdivide $\mathbb{Q}^d$ into orthant cones. Every point $x \in \mathbb{Q}^d$ belongs to exactly one polarized tangent cone according to the signs of the $n_i(x)$.

We can get from every polarized tangent cone to every other by successively flipping inequalities. Now the sum of two adjacent polarized tangent cones is defined by $d - 1$ (strict and non-strict) inequalities. Thus it is a cone that contains a line.   □

For a simple polytope, the indicator function version of Brion's formula (Theorem 1) can be derived from equation (4) (modulo cones that contain lines). As Theorem 1 does not specify these cones, there is no converse. On the other hand, generating functions do not see cones that contain lines. So for generating functions, the two formulas actually coincide.

The same considerations carry through for the weighted indicator functions. One thus obtains a weighted Brion's formula. We leave this as an exercise for the reader.

3. Non-simple polar decomposition

We want to generalize polar decomposition to non-simple polytopes. We will compute the local contribution from a simple deformation of the vertex in question, and prove that the result does not depend on the chosen deformation.

As there are not too many non-simple polytopes in dimension ≤ 2, the new running example will be the pyramid with the five vertices $(0, 0, 0)$, and $(\pm 1, \pm 1, 1)$.

Let $n_1, \ldots, n_N$ be the inner facet normals to P at v. They generate the rays of the normal cone $\mathcal{N}_v$. A virtual deformation of the vertex v is a regular triangulation[8] Δ of $\mathcal{N}_v$. That is a face-to-face subdivision of

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[8]Compare [**Lee97**, § 14.3]. Moving the facets according to the values of the convex function at the n_i would yield an actual simple deformation of the vertex v.
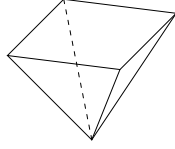
FIGURE 3. The simplest non-simple polytope.

$\mathcal{N}_v$ into simplicial cones $\sigma_1, \ldots, \sigma_M$, so that there is a convex piecewise linear function on $\mathcal{N}_v$ with domains of linearity the σ_i.
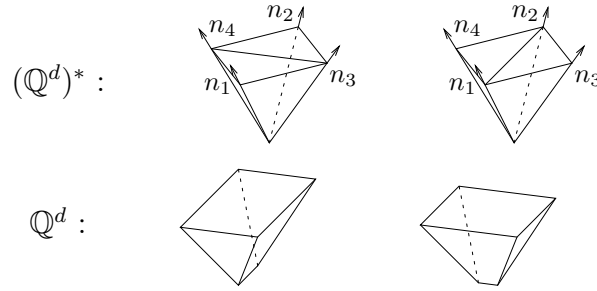

FIGURE 4. Triangulations and the corresponding deformations.

In the example, $v = (0, 0, 0)$ is the only non-simple vertex. The normal cone is generated by $n_1 = (1, 0, 1)$, $n_2 = (-1, 0, 1)$, $n_3 = (0, 1, 1)$, and $n_4 = (0, -1, 1)$. It has two triangulations, both of which are regular: Δ_1 with maximal simplices spanned by n_1, n_3, n_4 and n_2, n_3, n_4, and Δ_2 with maximal simplices spanned by n_1, n_2, n_3 and n_1, n_2, n_4. The corresponding deformations are sketched in Figure 4.

Now the tangent cone $\mathcal{T}_v P$ can be written as the intersection of simple cones defined by the σ_i:

$$(6) \quad \mathcal{T}_v P = \bigcap_i \mathcal{T}_{\sigma_i} \text{ for } \mathcal{T}_{\sigma_i} = \{x \in \mathbb{Q}^d : n_j(x) \geq n_j(v) \text{ for all } n_j \in \sigma_i\}.$$

For generic[9] $\xi \in (\mathbb{Q}^d)^*$, we write ξ in terms of the $n_j \in \sigma_i$, and flip those inequalities in (6) where ξ has negative coefficients. Then we compute the index $\mathrm{ind}_\xi(\sigma_i)$ as the number of flipped inequalities, and define the local contribution at v as:

$$\mathbb{1}_{\Delta, v}^\xi = \sum_i (-1)^{\mathrm{ind}_\xi(\sigma_i)} \mathbb{1}_{\mathcal{T}_v^\xi \mathcal{T}_{\sigma_i}}$$

The cool thing is that $\mathbb{1}_{\Delta, v}^\xi = \mathbb{1}_v^\xi$ does not depend on the triangulation. So it really is a *local* contribution.

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[9]Now, generic means that ξ is not constant on any ray of any of the $\mathcal{T}_{\sigma_i}$'s, i.e., ξ does not lie on any hyperplane used in the triangulation.

THEOREM 4. *Let $P \subset \mathbb{Q}^d$ be any polytope, and let $\xi \in (\mathbb{Q}^d)^*$ be generic.*
Then $\mathbb{1}^\xi_{\Delta,v} = \mathbb{1}^\xi_v$ does not depend on the choice of a regular triangulation
Δ of the inner normal cone $\mathcal{N}_v$ at vertex v. Moreover,

$$(7) \qquad\qquad \mathbb{1}_P(x) = \sum_{v \text{ vertex of } P} \mathbb{1}^\xi_v(x).$$

The Δ-invariance can be shown, e.g., using the fact that all regu-
lar triangulations are connected by flips [**Lee97**, § 14.6]. In this note,
however, we follow a different strategy. First, we will show the decom-
position formula (7) for compatible choices of triangulations (Lemma
5). Then, we observe that under relatively weak conditions, all decom-
positions are the same (Lemma 6).

But let us first see how the Δ-invariance works out in our example.
If we choose $\xi = (4, 2, 0)$, then we compute for triangulation Δ_1 as
follows: $\xi = 4n_1 - n_3 - 3n_4 = -4n_2 + 3n_3 + n_4$. Thus the polarized
cones are

$$\mathcal{T}^\xi_v \mathcal{T}_{\sigma_1} = \{x \ : \ n_1(x) \ge 0, n_3(x) < 0, n_4(x) < 0\} \text{ with index 2 and}$$

$$\mathcal{T}^\xi_v \mathcal{T}_{\sigma_2} = \{x \ : \ n_2(x) < 0, n_3(x) \ge 0, n_4(x) \ge 0\} \text{ with index 1.}$$

The trace of $\mathbb{1}_{\mathcal{T}^\xi_v \mathcal{T}_{\sigma_1}} - \mathbb{1}_{\mathcal{T}^\xi_v \mathcal{T}_{\sigma_2}}$ in the $\xi = 12$ plane is sketched in Figure 5
on the left. Similarly, for Δ_2 we get

FIGURE 5. The local contribution from Δ_1 and from Δ_2.

$$\mathcal{T}^\xi_v \mathcal{T}_{\sigma_3} = \{x \ : \ n_1(x) \ge 0, n_2(x) < 0, n_3(x) \ge 0\} \text{ with index 1 and}$$

$$\mathcal{T}^\xi_v \mathcal{T}_{\sigma_4} = \{x \ : \ n_1(x) \ge 0, n_2(x) < 0, n_4(x) < 0\} \text{ with index 2.}$$

The trace of $\mathbb{1}_{\mathcal{T}^\xi_v \mathcal{T}_{\sigma_4}} - \mathbb{1}_{\mathcal{T}^\xi_v \mathcal{T}_{\sigma_3}}$ in the $\xi = 12$ plane is sketched in Figure 5
on the right. Observe that $\mathcal{T}^\xi_v \mathcal{T}_{\sigma_3}$ and $\mathcal{T}^\xi_v \mathcal{T}_{\sigma_4}$ cancel out on the overlap
to yield the same contribution as $\mathcal{T}^\xi_v \mathcal{T}_{\sigma_1}$ and $\mathcal{T}^\xi_v \mathcal{T}_{\sigma_2}$.

LEMMA 5. *Let $P \subset \mathbb{Q}^d$ be any polytope, and let $\xi \in (\mathbb{Q}^d)^*$ be generic. Suppose Δ is a regular triangulation of the polar dual polytope P^*. It restricts to regular triangulations Δ_v of the normal cones $\mathcal{N}_v$. Then*

$$(8) \qquad \mathbb{1}_P(x) = \sum_{v \ vertex \ of \ P} \mathbb{1}^{\xi}_{\Delta_v, v}(x)$$

One can mimic the proof for the simple version (4) in order to show that the sum of the $\mathbb{1}^{\xi}_{\Delta_v, v}$ over all vertices of P does not depend on ξ, and that for every $x \in \mathbb{Q}^d$ there is a suitable ξ such that equation (8) is satisfied.

Now we actually use (8) together with the following lemma in order to show that $\mathbb{1}^{\xi}_{\Delta, v} = \mathbb{1}^{\xi}_v$ is independent of the triangulation.

LEMMA 6. *Let $P \subset \mathbb{Q}^d$ be any polytope, and let ξ be a generic element of $(\mathbb{Q}^d)^*$. Suppose that there are two decompositions*

$$\mathbb{1}_P \ = \sum_{v \ vertex \ of \ P} f_v \ = \sum_{v \ vertex \ of \ P} g_v$$

that are both conic: *for every direction $t \in \mathbb{Q}^d \setminus 0$, the $f_v(v + \lambda t)$ and $g_v(v + \lambda t)$ are constant in $\lambda > 0$, and* positive: *$f_v(v + t) = g_v(v + t) = 0$ if $\xi(t) < 0$. Then $f_v = g_v$.*
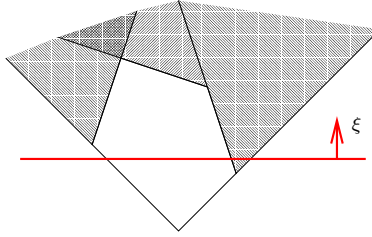

FIGURE 6. Positive conic decomposition of $\mathbb{1}_P$ is unique.

PROOF. This proof illustrates what was meant by Morse theory earlier on. We sweep over the polytope bottom to top, and only at the vertices does something happen. That is, we compare the restrictions of the functions to the hyperplanes $\xi = c$. Order the vertices $v_1, \ldots, v_k$ in ξ-increasing order. For $c \in [\xi(v_i), \xi(v_{i+1})[$, we have

$$g_{v_i} \overset{(\mathbf{p})}{=} \sum_{j=i}^{k} g_{v_j} = \mathbb{1}_P - \sum_{j=1}^{i-1} g_{v_j} \overset{(\mathbf{c})}{=} \mathbb{1}_P - \sum_{j=1}^{i-1} f_{v_j} = \sum_{j=i}^{k} f_{v_j} \overset{(\mathbf{p})}{=} f_{v_i}$$

along $\xi = c$. For $(\mathbf{p})$ we use positivity of the f_v's and g_v's, and for $(\mathbf{c})$ we use induction, and the fact that the f_v's and g_v's are conic. $\qquad \square$

Another incarnation of *the* positive conic decomposition is to group the summands in the Brianchon-Gram formula (1) as we did in equation (5), though it is less obvious that this is positive.

3.1. Homework. *or* **Why this is no good.** This is the wild speculation section. At the risk of exposing the full extent of my ignorance, I ask a bunch of questions that I stumbled over while compiling these notes.

If a good proof is one that makes us wiser [**Man77**], then this is a bad proof. It would be nicer to have a deformation independent definition of the local contribution in terms of the facet hyperplane arrangement at each vertex. How much geometry is needed? Can we compute the local contribution in purely combinatorial terms?

In some sense, the actual deformation of a vertex into several simple vertices converges to the non-simple vertex, and the local contribution is continuous. While this asks for a topology on the polytope algebra, I believe that this is a more combinatorial question. There must be a combinatorial deformation theory for (Eulerian?) posets.

Where are Gröbner bases? Often when regular triangulations show up, the corresponding convex function determines a term order. Does the result tell us something about commutative algebra?

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