LATTICE POLYTOPES IN ALGEBRA, PHYSICS, AND OPTIMIZATION.

RESEARCH REPORT OF THE EMMY NOETHER OF THE RESEARCH GROUP LATTICE POLYTOPES 2005-2008

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1.2. **Publications of the Group.**

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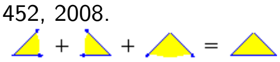
[7] Matthias Beck, Christian Haase and Frank Sottile.  (Theorems of Brion, Lawrence, and Varchenko on rational generating functions for cones). *American Mathematical Monthly*, to appear, `math.CO/0506466`.
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2. Research Report

This is a report on the results obtained by the Emmy Noether Research Group "Lattice Polytopes" during the funding period 7/2005-12/2008. During this period, Christian Haase used the possibilities offered by the Emmy Noether grant to build the group, winning Benjamin Nill from Tübingen and Andreas Paffenholz from TU Berlin. These two excellent junior mathematicians incorporated their own profile into the work of the group, and the intended research program could be tackled fairly quickly and successfully.

Most notably, a boundedness result of Lagarias/Ziegler was generalized to polytopes without interior lattice points (see §2.3 The Degree of Lattice Polytopes). Degree bounds (perhaps surprisingly low) could be provided for generators of certain toric ideals (see §2.4 Toric Gröbner Bases). The study of permutation polytopes (see §2.5) was not anticipated in the original proposal. It grew out of hallway discussions with the local group theorist Barbara Baumeister.

While the group was able to deal with most of the projected program, the question for necessary conditions for the existence of unimodular triangulations and related properties proved to be even harder than expected. To cope with these difficulties, the group joined forces with other researchers during an Oberwolfach Mini-Workshop in 2007 (and a follow-up at AIM in 2009). With the development of a new `polymake` lattice point environment (see §2.7), and with the help of student assistant Benjamin

Müller, the Research Group should be on track to experiment and to find interesting examples.

The results in this report are grouped into five categories: Ehrhart Theory, Unimodular Triangulations, Permutation Polytopes, Mirror Symmetry, and Polymake. In each section the relevant notions are briefly introduced, and the most important findings are described. Ongoing research and future directions are not treated here, but in the attached proposal for an extension. Let us start with an introduction into the field.

2.1. **Field of Research.** How many non–negative integral solutions does the following equation have?[1]

$$12\,223x_1 + 12\,224x_2 + 36\,674x_3 + 61\,119x_4 + 85\,569x_5 = 89\,643\,482$$

Questions like the above have applications in a wealth of areas outside mathematics. At the same time, they appear in different disguises in various mathematical fields. For example, the original question has a number theoretical flavor. But in view of a discrete geometer it actually asks for the number of lattice points in a polyhedron. In commutative algebra one would ask for the Hilbert series of a graded ring, and in algebraic geometry for the Todd class of a toric variety. The (apparently simpler) question whether there is a solution at all is an integer linear optimization (knapsack) problem.

Single interactions between these disciplines have been explored in the past. Classic geometry of numbers was an immensely successful application of (discrete) geometric methods to number theory around a century ago [GL87]. Enumeration of solutions of Diophantine equations has stimulated interactions between commutative algebra and enumerative combinatorics since the 70's [Sta96, BH93]. The theory of toric varieties (algebraic geometry & discrete geometry) has developed into an established and very active field of research [Dan78, Ewa96, Ful93, Oda88]. More recently, we have seen applications of Gröbner bases to optimization problems – and vice versa (computational algebra & integer programming) [HT98, Stu96, MS05], Dedekind sums in lattice point problems (number theory & discrete geometry) [BP99, Pom93], and ideal theoretic investigations of polytopes (commutative algebra & discrete geometry) [BGT97]. Lattice polytopes have been "discovered" by representation theorists (Kostka and Littlewood-Richardson coefficients) [BZ88, BZ01, KT99], theoretical physicists (Mirror symmetry) [Bat94, BD96, BB96], and statisticians (Contingency tables) [DS98, CDDH05].

2.2. **Background and Notation.** In what follows, P will be a lattice polytope. That is, P is the convex hull in $\mathbb{R}^n$ of finitely many points in the lattice $\mathbb{Z}^n$. We will identify two lattice polytopes if they are related by a lattice preserving affine map. Up to this lattice equivalence, we can always assume that our polytope is n-dimensional. It is often convenient to consider the cone $\sigma_P \subset \mathbb{R}^{n+1}$ generated by $P \times \{1\}$. For more on convex polytopes and lattices we refer to [Bar02].

A unimodular simplex is a lattice polytope which is lattice equivalent to the standard simplex: the convex hull of the origin $\mathbf{0}$ together with the standard unit vectors $\boldsymbol{e}_i$ $(1 \leq i \leq n)$. Equivalently, unimodular simplices are characterized as the n-dimensional lattice polytopes of minimal possible Euclidean volume $1/n!$.

[1] From [AL02]. There is a unique solution, but commercial IP solvers need hours to compute it [LHH+04].

Many properties of the combinatorial objects have direct translations to algebraic objects like semigroup algebras, monomial ideals and toric varieties, via the simple correspondence

$$\begin{array}{ccc} \text{lattice point} & & \text{Laurent monomial} \\ \boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{Z}^n & \longleftrightarrow & \boldsymbol{x}^{\boldsymbol{v}} := x_1^{v_1} \cdot \ldots \cdot x_n^{v_n} \in \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]. \end{array}$$

Algebraic geometers and commutative algebraists are interested in the properties of the graded semigroup ring $R_P = \Bbbk[\sigma_P \cap \mathbb{Z}^{n+1}]$ and the corresponding projective toric variety $X_P = \operatorname{Proj} R_P$.

2.3. **Ehrhart Theory.** Many counting problems can be phrased as counting lattice points in (dilates) of polytopes or polyhedral complexes [Loe05, BR07a]. By a fundamental result of Eugène Ehrhart, the number of lattice points in positive dilates kP of P is a polynomial function in $k \in \mathbb{Z}_{>0}$. (See [Ehr67, BR07a].) Consequently, the generating function – the Hilbert function of R_P – has a special form.

$$\operatorname{Hilb}_{R_P}(t) = \sum_{\boldsymbol{v} \in \sigma_P \cap \mathbb{Z}^{n+1}} t^{v_{n+1}} = \sum_{k \geq 0} \#(kP \cap \mathbb{Z}^n) \; t^k \;=\; \frac{h^*(t)}{(1-t)^{n+1}}\,,$$

with a polynomial $h^* \in \mathbb{Z}_{\geq 0}[t]$ of degree $d \leq n$ [Sta93, BS07].

The Degree of Lattice Polytopes. The degree d of h^* is called the degree of P. The polytope P has no interior lattice points if and only if the degree of P is strictly smaller than n. Moreover, the smaller the degree the more multiples of P contain no interior lattice points. The fundamental goal with respect to this new invariant is to find a classification of lattice polytopes with given degree. This is known for degrees 0 and 1 [BN07].

In joint work with Sam Payne [HNP09], Christian Haase and Benjamin Nill could show that a small degree implies strong geometric and structural consequences: P splits into $n - (d^2 + 19d - 4)/2$ Cayley factors. This means that P is equivalent to the convex hull of $(P_1, \boldsymbol{e}_1), \ldots, (P_r, \boldsymbol{e}_r) \subset \mathbb{R}^{s+r}$ for polytopes $P_1, \ldots, P_r \subset \mathbb{R}^s$ with $s \leq (d^2 + 19d - 4)/2$. This result answers positively a question of Benjamin Nill and Victor Batyrev in [BN07]. The proof is based on an observation [Nil08] about lattice polytopes of fixed degree and linear coefficient h_1^*, which also includes a purely combinatorial strengthening of a recent theorem of Batyrev [Bat07].

As an application, Haase/Nill/Payne prove the following conjecture of Batyrev [Bat07]: up to pyramid constructions, there is only a finite number of lattice polytopes of given degree and leading coefficient h_d^*. This is a vast generalization of the theorem of Douglas Hensley and Jeffrey Lagarias/Günter Ziegler which says that in fixed dimension there are, up to lattice equivalence, only finitely many polytopes with given positive number of interior lattice points [Hen83, LZ91].

There is a close relation of these results to adjunction theory of toric varieties. In the intensively studied field of adjunction theory the main question is to classify polarized varieties X having large nef-value, namely, where an ample line bundle $\mathcal{L}$ has to be multiplied by a large multiple k so that $K_X + k\mathcal{L}$ is nef. Continuing [HNP09], this relation was further investigated by Alicia Dickenstein, Sandra Di Rocco, and Ragni Piene in a new preprint [DRP08], where they could sharpen the bounds from order $O(d^2)$ to $2d + 1$ for certain smooth toric varieties.

Lattice Point Generating Functions. The lattice point $\leftrightarrow$ Laurent monomial correspondence in the introduction allows us to encode the set of lattice points in the polytope P or in a pointed rational cone σ by the sum of the corresponding monomials: $S_P(\boldsymbol{x}) = \sum_{\boldsymbol{v} \in P \cap \mathbb{Z}^n} \boldsymbol{x}^{\boldsymbol{v}}$ is a Laurent polynomial, and $S_\sigma = \sum_{\boldsymbol{v} \in \sigma \cap \mathbb{Z}^{n+1}} \boldsymbol{x}^{\boldsymbol{v}}$ can be interpreted as a rational function [BR07a]. Alexander Barvinok used this encoding in his acclaimed polynomial time lattice point counting algorithm in fixed dimension [Bar94a, Bar94b]. Matthias Beck, Christian Haase and Frank Sottile review elementary proofs of structural results of Brion and of Lawrence-Varchenko on these lattice-point enumerator generating functions S_P and S_σ [BHS06].

The generating functions also have applications in number theory. James Pommersheim introduced Dedekind sums in Ehrhart theory [Pom93]. Matthias Beck, Christian Haase and Asia Matthews [BHM08] study higher-dimensional analogs of Dedekind-Carlitz polynomials $\mathrm{Carlitz}\,(u, v; a, b) := \sum_{k=1}^{b-1} u^{\left\lfloor \frac{ka}{b} \right\rfloor} v^{k-1}$, where u and v are indeterminates and a and b are positive integers. Carlitz proved that these polynomials satisfy a reciprocity law from which one easily deduces many classical reciprocity theorems for the Dedekind sum and its generalizations. Beck/Haase/Matthews illustrate that Dedekind-Carlitz polynomials appear naturally in lattice point generating functions and use this fact to give geometric proofs of the Carlitz reciprocity law and various extensions of it. Their approach gives rise to new reciprocity theorems and computational complexity results for Dedekind-Carlitz polynomials.

2.4. **Unimodular Triangulations.**

A face-to-face subdivision of P into unimodular simplices is a unimodular triangulation. A triangulation is called regular (sometimes also called projective or coherent) if the simplices are the domains of linearity of a convex function. Presumably, polytopes that admit a unimodular triangulation are very rare.

Projective Normality. There is an entire network of related combinatorial and algebraic properties of polytopes P and cones σ (cf. the hierarchy [MFO07, p.2313]). Christian Haase, Diane Maclagan and Takayuki Hibi organized an Oberwolfach Mini-Workshop on the subject. As a direct result of this meeting Christian Haase, Benjamin Nill, Andreas Paffenholz and Francisco Santos were able to settle the ample+nef question for general (singular) toric surfaces [HNPS08]. Translated back into polyhedral language, the authors showed that for two lattice polygons P and Q any lattice point in their Minkowski sum can be written as a sum of a lattice point in P and one in Q, provided the normal fan of P is a subdivision of the normal fan of Q. The same question is open in dimension 3, even if we assume that $P = Q$ describes a smooth toric variety $X_P = \mathrm{Proj}\, R_P$.

Toric Gröbner Bases. There is a close connection between the Gröbner bases of the defining ideal I_P of $R_P = \Bbbk[x_1 \ldots x_N]/I_P$ ($N = |P \cap \mathbb{Z}^n|$) on the one hand, and the regular triangulations of P on the other [GKZ94, Stu96]. Kapranov, Sturmfels and Zelevinsky show that the ideal I_P has a squarefree initial ideal if and only if P admits a regular unimodular triangulation [KSZ92]. In this case, the corresponding Gröbner basis can be read off from the triangulation. In this context, there is the folklore conjecture that the ideal I_P should be generated by quadratics or even have a quadratic Gröbner basis if the projective toric variety $X_P = \mathrm{Proj}\, R_P$ is smooth (cf. the discussion by Allen Knutson [BCF+05, p.186f]).

Families of such smooth (sometimes also called regular or Delzant) polytopes are given by generic transportation polytopes and by smooth reflexive polytopes (see §2.6). Christian Haase and Andreas Paffenholz developed techniques to construct (quadratic) toric Gröbner bases and applied them to these classes.

- Contrary to previous expectations, Haase/Paffenholz showed that I_P does admit a quadratic Gröbner basis if P is a smooth 3×3 transportation polytope [HP09]. This should be contrasted by the fact that the ideal of the 3×3 Birkhoff polytope – a non-smooth transportation polytope – is *not* quadratically generated. Matthias Lenz [Len07] in his Master's thesis was able to adapt the method to show that the toric ideals of all flow polytopes, smooth or not, are generated in degree three. (Flow polytopes are the natural generalization of transportation polytopes. They can be realized as faces of transportation polytopes.) In the case of the $n \times n$ Birkhoff polytope this was conjectured by Diaconis and Eriksson [DE06, Conj. 7].
- Smooth reflexive polytopes have been classified up to dimension 8 (cf. §2.6). Haase/Paffenholz used this classification and a project-and-lift heuristic to construct quadratic Gröbner bases for most of the polytopes in the data base [HP07].

2.5. **Permutation Polytopes.** This project grew out of a weekly joint seminar with Barbara Baumeister during the 2005/6 academic year.

A permutation polytope is the convex hull in $\mathbb{R}^{n \times n}$ of a subgroup of the $n \times n$ permutation matrices. An important example is given by the Birkhoff polytope: the convex hull of all permutation matrices. This is an extensively studied polytope appearing in many different fields of mathematics (see e.g. Brualdi and Gibson [BG77a, BG77b, BG77c] and Billera and Sarangarajan [BS96]).

In contrast, general permutation polytopes have attracted little attention in the past. A number of authors have studied special classes: Richard Brualdi and Bolian Liu compute basic invariants of the polytope of the alternating group [BL91]; for this polytope, Jeffrey Hood and David Perkinson [HP04] describe exponentially many facets. John Collins and David Perkinson [CP04] observe that Frobenius polytopes have a particularly simple combinatorial structure, and Hana Steinkamp [Ste99] adds results about dihedral groups. Most recently, Robert Guralnick and David Perkinson [GP06] investigate general permutation polytopes, their dimension, and their graph from a group theoretic viewpoint.

The aim of this project is to extend known results on the Birkhoff polytope to the setting of permutation polytopes, or even more general, representation polytopes. The driving hope is to use polyhedral methods to derive results in representation theory of permutation groups. This has turned out to be quite successful. Still, many unsolved questions are open which can be tackled from convex geometric, computational or group-theoretic perspectives. In a joint paper with Barbara Baumeister, Christian Haase, Benjamin Nill and Andreas Paffenholz gave a first account of their systematic study of permutation polytopes [BHNP09].

- The authors clarify the relevant notions of equivalence of group representations: the polytope does not reflect multiplicities of irreducible representations or automorphisms of the group.
- They show that if a permutation polytope is combinatorially a product, then the permutation group has a natural product structure.

- They classify centrally symmetric permutation polytopes via certain subspaces of $\mathbb{F}_2^r$. In particular, the group has to be an elementary abelian 2-group, and the number of vertices has to be a power of 2.
- They use combinatorial properties of (faces of) permutation polytopes to classify ≤ 4-dimensional permutation polytopes and the corresponding permutation groups.
- They give a list of open questions and conjectures.

2.6. **Mirror Symmetry.** The mirror symmetry conjecture in string theory predicts that Calabi-Yau varieties come in pairs $(X, X^\vee)$ which determine "the same physics" [Mor97, CK99]. One consequence of this statement is the fact that the stringy Hodge numbers of X and $X^\vee$ satisfy a mirror symmetry.

Nef-partitions. There is a general framework developed by Victor Batyrev and Lev Borisov for the construction of mirror pairs using toric geometry. The key player is the notion of a Gorenstein polytope. We call a lattice polytope reflexive if its polar dual with respect to an interior lattice point is again integral. Then, a lattice polytope P so that the dilation rP is reflexive for some $r \in \mathbb{Z}_{>0}$ is a Gorenstein polytope of index r. Some Gorenstein polytopes admit nice decompositions – nef-partitions – giving rise to Calabi-Yau complete intersections. This additional structure allows a combinatorial duality extending polar duality for reflexive polytopes. The Calabi-Yau's obtained this way satisfy the Hodge number test [BB96, BD96].

Benjamin Nill and Victor Batyrev clarify precisely how nef-partitions fit into this setting [BN08]. For this, they consider the duality between two nef-partitions as a duality between two Gorenstein polytopes P and P^* of index r together with selected special $(r-1)$-dimensional simplices S in P and S' in P^*. Different choices of these simplices suggest an interesting relation to Homological Mirror Symmetry.

Recently, this notion of a special simplex has also been prominently used by Christos Athanasiadis to prove Stanley's conjecture on the unimodality of the h^*-vector of Birkhoff polytopes [Ath05]. In this context Batyrev/Nill also give a simpler combinatorial proof of a result of Winfried Bruns and Tim Römer on projectively normal Gorenstein polytopes [BR07b].

Beyond stringy Hodge numbers, mirror symmetry yields predictions about the the fundamental groups and integral cohomologies of Calabi-Yau's. In their combinatorial computation, lattices generated by lattice points in skeletons of reflexive polytopes play an essential role. Christian Haase and Benjamin Nill prove that the lattice generated by all lattice points in a reflexive polytope of dimension ≥ 3 is already generated by lattice points in codimension two faces [HN08]. This answers a question of J. Morgan.

Toric SYZ. Strominger, Yau and Zaslow proposed an interpretation of mirror symmetry as a duality of special Lagrangean torus fibrations [SYZ96] (compare also [Mor99]). In the reflexive polytope case, Christian Haase and Ilia Zharkov had previously described a combinatorial model for the conjectured integral affine structure on the base of the fibration. The model has mirror symmetry built in [HZ02, HZ03]. Now, Haase/Zharkov have extended their model to the case of nef-partitions. They prove that this base space is topologically a product of spheres [HZ05].

Reflexive Dimension. Besides the original physical motivation, reflexive polytopes are increasingly emerging as useful tools for algebraic applications in combinatorics (and vice versa). There is a close relation to polytopal adjunction theory discussed in §2.3. In fixed dimension, there are only finitely many reflexive and Gorenstein polytopes [KS98, KS00]. On the other hand, Christian Haase and Ilarion Melnikov show that every given lattice polytope P appears as a face of some, possibly high dimensional, reflexive polytope [HM06]. Call the smallest dimension of such a reflexive polytope the reflexive dimension $\operatorname{refldim}(P)$ of P. Then, Haase/Melnikov use a number theoretic result of Michael Vose on Egyptian fractions [Vos85] to give bounds on the reflexive dimension of dilations:

$$m \log\log k \leq \operatorname{refldim}(kP) - \operatorname{refldim}(P) \leq M\sqrt{\log k}$$

for constants m, M.

Toric Fano Varieties. A natural generalization of reflexive polytopes are Fano polytopes: lattice polytopes that contain the origin in their interiors and whose vertices are primitive. In algebraic geometry, Fano varieties play an especially important role, e.g., in the Minimal Model Program. It is desirable to find classification results and bounds on their invariants. Three years ago, Cinzia Casagrande [Cas06] proved $\rho_X \leq 2n$ for the Picard number ρ_X of an n-dimensional $\mathbb{Q}$-factorial Gorenstein toric Fano variety X. She also classified the $\rho_X = 2n$ case. Benjamin Nill and Mikkel Øbro [Øbr08, NØ08] extended this result by giving a complete list for $\rho_X = 2n-1$. Benjamin Nill and Maximilian Kreuzer gave a classification of ≤ 5-dimensional nonsingular toric Fano varieties. Later, Mikkel Øbro continued independently up to dimension 8 [Øbr07a, Øbr07b]. In dimension two, (singular) Fano varieties are called log Del Pezzo surfaces. Benjamin Nill and Dimitrios I. Dais [DN08] could improve in the toric case a theorem due to Viacheslav V. Nikulin [Nik89], which bounds the Picard number in terms of the index ℓ_X of the log Del Pezzo surface X. Alexander Kasprzyk, Benjamin Nill and Maximilian Kreuzer [KKN08] provided cubic bounds on the volume of the corresponding Fano polygons, efficient classification algorithms and complete lists for $\ell_X \leq 16$.

2.7. **Polymake.** The software `polymake` [GJ, GJ05] is a general framework for computations with polyhedra. Given a polyhedron defined by points or inequalities, `polymake` works with a large set of rules that specify how new properties may be derived from known ones. `polymake` comes with several clients that compute new properties according to the rules, but it may also call external software, if available. `polymake` has been developed at TU Berlin by Michael Joswig and Ewgenij Gawrilow since 1997. Its current version is 2.3.

Benjamin Müller and Andreas Paffenholz have written a new set of rules that enables `polymake` to compute properties that are specific for lattice polytopes, like the number of lattice points inside a polytope, or its h^*-vector. These new properties will be contained in the upcoming release 3 of `polymake`.

This project had two parts:

- Interfaces to existing software.

  There already exist software packages that can compute various properties of lattice polytopes. Müller/Paffenholz have implemented interfaces that call the appropriate program for the production of these properties. So far, the following programs can be used:

  – `normaliz` [BK, BK01] computes Hilbert bases, h^*-polynomials, and volumes.

– LattE [LHYT, LHTY04, Köp], computes generating functions for lattice points in a polytope, and derives the number of lattice points and the Ehrhart polynomial.
– 4ti2 [4ti] can solve systems of equations over $\mathbb{Z}$, and uses this to compute Hilbert bases of cones and Gröbner bases of the corresponding toric ideals.

- Rules that do not depend on external software.

  Müller/Paffenholz have implemented rules for several important properties of lattice polytopes that are not implemented in any other software package. This e.g. includes the possibility to decide whether a polytope is reflexive or smooth, or the computation of its facet width.

  Many of these properties may be obtained in various different ways. This is reflected in the implementation by rules that choose an efficient way of computing a property depending on what data is already known.

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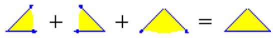
[BHS06] Matthias Beck, Christian Haase, and Frank Sottile.  (theorems of brion, lawrence, and varchenko on rational generating functions for cones). *American Mathematical Monthly*, 2006. to appear; `math.CO/0506466`.

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2.8. **Personnel.**

Dr. Christian Haase	(Gruppenleiter, Emmy Noether)	18.07.07 – 17.07.09
Dr. Benjamin Nill	(Postdoc, Emmy Noether)	30.09.05 – 17.07.09
Dr. Andreas Paffenholz	(Postdoc, Emmy Noether)	01.10.05 – 31.03.07
	(Assistent, FU Berlin)	01.04.07 – 17.07.09
Frederik von Heymann	(Mitarbeiter, Emmy Noether)	01.12.07 – 31.01.08
	(Mitarbeiter, FU Berlin)	01.02.08 – 24.10.08
	(Mitarbeiter, Emmy Noether)	25.10.08 – 31.03.09
Thomas Friedrich	(Mitarbeiter, Emmy Noether)	15.10.08 – 14.01.09
Benjamin Müller	(Stud. Hilfskraft, Emmy Noether)	01.03.08 – 17.07.09
Matthias Lenz	(Diplomand)	15.10.05 – 31.07.07

2.9. **Cooperations.**

Victor Batyrev, Tübingen (mirror symmetry, h^*-degree)
Barbara Baumeister, FU Berlin (permutation polytopes)
Matthias Beck, San Francisco State (Ehrhart theory, number theory)
Felix Breuer, FU Berlin (graph polynomials)
Dimitrios Dais, Univ. of Crete (Fano polytopes, mirror symmetry)
Ewgenij Gawrilow, TU Berlin (`polymake`)
Michael Joswig, TU Darmstadt (`polymake`)
Alexander Kasprzyk, Univ. of Kent (Fano polytopes)
Maximilian Kreuzer, TU Wien (reflexive polytopes)
Asia Matthews, Queen's University (Ehrhart theory, number theory)
Tyrrell McAllister, MPI Bonn ($\mathbb{Z}$-scissors congruence, representation theory)
Ilarion Melnikov, MPI Golm (mirror symmetry, reflexive dimension)
Mikkel Øbro, Aarhus (reflexive polytopes)
Sam Payne, Stanford (Ehrhart theory, toric algebra)
Francisco Santos, U de Cantabria (unimodular triangulations)
Josef Schicho, Linz (polytopal adjunction theory, toric algebra)
Frank Sottile, Texas A&M (Ehrhart theory, toric algebra)
Ilia Zharkov, Kansas (mirror symmetry, tropical j-invariant)

2.10. **Conferences Attended.**

▶ Christian Haase:

- Darmstadt, "A Saturday on Discrete Mathematics" (26. - 27. November 2005)
- Oberwolfach, "Convex and Algebraic Geometry" (29. Januar - 4. Februar 2006)
- Linz, RICAM, "Gröbner Bases in Cryptography, Coding Theory, and Algebraic Combinatorics" (30. April - 7. Mai 2006)
- Osnabrück, "Kommutative Algebra" (2. - 3. Juni 2006)
- Snowbird, USA, "Integer Points in Polyhedra" (9. - 18. Juni 2006)
- Alcalá de Henares, Spanien, "Workshop on Geometric and Topological Combinatorics" (31. August - 6. September 2006)
- Oldenburg, "Tag der Mathematik" (8. November 2006)
- Oldenburg, Colloquium (16. - 17. November 2006)
- Kyoto, RIMS, "Theoretical Effectivity and Practical Effectivity of Gröbner Basis" (20. - 27. Januar 2007)
- Darmstadt, Optimierungsseminar (26. Juni 2007)

- Univ. de Caen, Frankreich, "LLL25" (28. Juni - 1. Juli 2007)
- Oberwolfach, Mini–Workshop "Projective Normality of Smooth Toric Varieties" (12. - 18. August 2007)
- Warwick, "2007-08 Warwick EPSRC syymposium on Algebraic Geometry" (16. - 20. September 2007)
- Oberwolfach, Workshop "Tropical Geometry" (9. - 16. Dezember 2007)
- Frankfurt, "A Saturday on Discrete Mathematics" (25. - 27. Januar 2008)
- Magdeburg, Colloquium (15. Januar 2008)
- Aarhus, Member of PhD committee for Mikkel Øbro (10. - 12. Januar 2008)
- MPI Leipzig, Workshop "Geometric Aspects of Conditionsl Independence and Information" (13. - 14. März 2008)
- MPI Leipzig, Seminar day "Modellauswahl" (8. Mai 2008)
- Konstanz, Colloquium (9. - 10. Mai 2008)
- Stockholm, Conference "Festive Combinatorics in honor of Anders Bjoerner" (27. - 31. Mai 2008)
- Göttingen, "A tropical Wednesday in Göttingen" (9. - 10. Juli 2008)

▶ Benjamin Nill:

- Oberwolfach, Conference (29. Januar - 9. Februar 2006)
- Snowbird, USA, Conference "Integer Points in Polyhedra" (10. Juni - 18. Juni 2006)
- TU Wien, Kooperation mit Maximilian Kreuzer (26. Juni - 2. Juli 2006)
- Magdeburg, "Kolloquium über Kombinatorik"(16. - 18. November 2006)
- Istanbul, Sabanci Univ., Conference "GAeL" (17. - 23. Juni 2007)
- Oberwolfach, Mini–Workshop "Projective Normality of Smooth Toric Varieties" (12. - 18. August 2007)
- Univ. of Minnesota, Rutgers Univ., Stanford Univ., Kooperations- und Vortragsreise (30. Sept. - 14. Oktober 2007)
- Aarhus, Disputation Mikkel Øbro (10. - 12. Januar 2008)
- Univ. Göttingen, "A tropical Wednesday in Göttingen" (9. Juli 2008)
- Univ. of Catania, Workshop "P.R.A.G.MAT.I.C. 2008" (13. - 31. Juli 2008)

▶ Andreas Paffenholz:

- Snowbird, USA, Conference "Integer Points in Polyhedra" (10. Juni - 17. Juni 2006)
- Snowbird, USA, Conference "Discrete Geometry – 20 years later" (18. Juni - 24. Juni 2006)
- Alcalá de Henares, Spanien, Workshop on Geometric and Topological Combinatorics (30. August - 6. September 2006)
- Magdeburg, "Kolloquium über Kombinatorik"(16. - 18. November 2006)
- Berlin, DMV-Tagung (26. - 30. März 2007)
- Oberwolfach, Mini–Workshop "Projective Normality of Smooth Toric Varieties" (12. - 18. August 2007)
- Magdeburg, "Kolloquium über Kombinatorik"(16. - 17. November 2007)
- Erlangen, DMV-Tagung (14. - 19. September 2007)
- Magdeburg, "Kolloquium über Kombinatorik"(14. - 15. November 2008)

2.11. **Conference Organization.**

4.2009 Workshop *Combinatorial challenges in toric varieties* American Institute of Mathematics; funded by AIM, NSF; Christian Haase, Joseph Gubeladze and Diane Maclagan

4.2009 AMS Special Session *Algebra & Number Theory with Polytopes* in San Francisco; Matthias Beck and Christian Haase

1.2009 Arbeitsgemeinschaft "h^*-polynomials" at FU Berlin; Christian Haase, Benjamin Nill and Andreas Paffenholz

9.2008 Minisymposium at the DMV Annual Meeting in Erlangen; Christian Haase, Benjamin Nill and Andreas Paffenholz

4.2008– Seminar Days *Model Selection* in Berlin, Darmstadt, Leipzig, Magdeburg; Christian Haase, Raymond Hemmecke, Thomas Kahle and Alexander Schliep

4-7.2008 Research Seminar *Tropical Mathematics* at FU Berlin; Christian Haase, Walter Gubler, Michael Joswig and Bernd Sturmfels

8.2007 Mini-Workshop *Projective Normality of Smooth Toric Varieties* in Oberwolfach; Christian Haase, Takayuki Hibi and Diane Maclagan

5-7.2007 (Pre)Doc Program, FU Berlin; funded by Graduiertenkolleg *Methods for Discrete Structures*, Emmy Noether; Christian Haase

3.2007 Arbeitsgemeinschaft "Gorenstein and Reflexive Polytopes" at FU Berlin; Christian Haase, Benjamin Nill and Andreas Paffenholz

6.2006 Second AMS/IMS/SIAM Conference *Integer Points in Polyhedra* in Snowbird, UT; funded by NSF; Matthias Beck, et al.

8.2005 *Duke-Berlin Geometry and Physics Festival* FU Berlin; funded by NSF, SFB 647; Christian Haase and Sven Rinke

Berlin, December 10, 2008

Christian Haase