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INTEGRAL AFFINE STRUCTURES ON SPHERES AND TORUS FIBRATIONS OF CALABI-YAU TORIC HYPERSURFACES I

CHRISTIAN HAASE AND ILIA ZHARKOV

Abstract. We describe in purely combinatorial terms dual pairs of integral affine structures on spheres which come from the conjectural metric collapse of mirror families of Calabi-Yau toric hypersurfaces. The same structures arise on the base of a special Lagrangian torus fibration in the Strominger-Yau-Zaslow conjecture. We study the topological torus fibration in the large complex structure limit and show that it coincides with our combinatorial model.

1. Introduction

Since Strominger, Yau, and Zaslow conjectured an interpretation of mirror symmetry as a duality of special Lagrangian torus fibrations [SYZ96], there has been considerable progress toward proving the topological consequences of this conjecture. Recently, for a Calabi-Yau family in a neighborhood of the large complex structure point, Kontsevich and Soibelman [KS01] and Gross and Wilson [GW00] conjectured the existence of an affine Kähler structure on the (complement of a codimension two locus of the) limiting metric space of the Gromov-Hausdorff collapse. This metric space should be identified with the base of the SYZ fibration (with McLean's metric). In particular, the collapse picture asserts the existence of an integral affine structure on the base, which is enough to reconstruct the non-degenerate part of the topological torus fibration of the CY family.

An integral affine structure on the base of the conjectural SYZ torus fibration has been described by Ruan [Rua99] and Gross [Gro01] in the quintic case, followed by work of Ruan [Rua00] for general toric hypersurfaces. The present paper provides a short and explicit combinatorial description of an affine structure with mirror duality built in, based on an idea of David Morrison [Mor00].

Given a dual pair of d-dimensional reflexive polytopes with coherent triangulations of their boundaries, we construct in §2.1 a $(d-1)$-dimensional polytopal complex Σ, topologically a sphere, with a codimension 2 subcomplex D, which we call a discriminant locus. The manifold $\Sigma \backslash D$ possesses an integral affine structure (§2.3). That is, the tangent space at any point in $\Sigma \backslash D$ contains a natural integral lattice, and one can form a $(d-1)$-torus fibration $W \to \Sigma \backslash D$ by taking fiber wise quotients. The nerve of the covering of $\Sigma \backslash D$ by affine charts is a two-colored graph, whose nodes are labeled by the vertices of the triangulations and an edge connects any two which live in dual faces of the polytopes.

The more technical §2.2 is concerned with the sphericity of Σ. We develop a generalization of barycentric subdivisions which might be of independent interest to a combinatorially inclined reader.

In Section 3 we link the model to the topology of toric hypersurfaces H_s constructed from our input data (§3.1). The main result Theorem 3.7 asserts that for any neighborhood N of D, and a hypersurface with large enough complex structure, there is a torus fibration of H_s^{sm}, a portion of the hypersurface, over $\Sigma \backslash N$, which is diffeomorphic to the restriction of our model fibration.

In the second part of the paper we will develop a connection of our model to the *geometry* of the hypersurfaces, conjectured in [GW00] and [KS01]. Namely, we will show that the above diffeomorphism provides, in fact, an "almost" holomorphic embedding (there is a preferred choice of complex structures on W). Moreover, we will construct a family of Kähler forms on H_s in the expected class, so that the pairs $(H_s, H_s \backslash H_s^{\mathrm{sm}})$ with the induced metrics converge in the Gromov-Hausdorff sense to the pair (Σ, D).

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2. The combinatorial model

We are given a reflexive polytope $\Delta^\vee \subset \mathbb{R}^d$. That is, $\Delta^\vee$ is the convex hull of finitely many points in the lattice $\mathbb{Z}^d$, it contains the origin in the interior, and the vertices of the dual polytope $\Delta = \{m \in (\mathbb{R}^d)^* \ : \ \langle m, n\rangle \leq 1 \text{ for all } n \in \Delta^\vee\}$ belong to the dual lattice $(\mathbb{Z}^d)^*$ [Bat94].

Another part of our input are two sufficiently generic vectors $\lambda \in \mathbb{Z}^{\Delta \cap (\mathbb{Z}^d)^*}$, and $\nu \in \mathbb{Z}^{\Delta^\vee \cap \mathbb{Z}^d}$, which induce central coherent triangulations of Δ and $\Delta^\vee$. (Cf. [Lee97] for coherent/regular triangulations.) These triangulations restrict to triangulations S and T on the boundaries $\partial\Delta$ and $\partial\Delta^\vee$. Using λ and ν we define polytopes

$$\Delta_\nu = \{m \in (\mathbb{R}^d)^* \ : \ \langle m, n\rangle \leq \nu(0) - \nu(n) \text{ for all } n \in \Delta^\vee \cap \mathbb{Z}^d\}$$
$$\Delta^\vee_\lambda = \{n \in \mathbb{R}^d \ : \ \langle m, n\rangle \leq \lambda(0) - \lambda(m) \text{ for all } m \in \Delta \cap (\mathbb{Z}^d)^*\},$$

whose normal fans are given by the cones spanned by the faces of T respectively S.

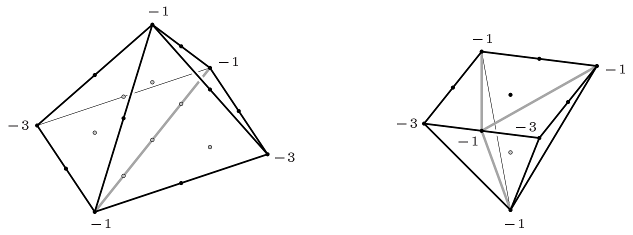


FIGURE 1: $\Delta = \operatorname{conv}\begin{bmatrix} 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 1 & -1 & -1 & -1 & -1 \end{bmatrix}$ $\quad \Delta^\vee = \operatorname{conv}\begin{bmatrix} 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \end{bmatrix}$

The values of λ and ν are marked on the vertices, $\lambda(0) = \nu(0) = 0$.

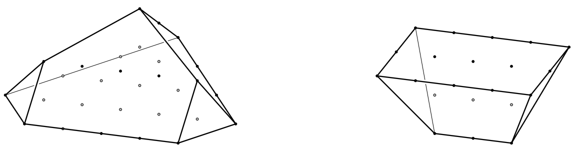


FIGURE 2: Δ_ν $\qquad \Delta^\vee_\lambda$

2.1. **The base and the discriminant locus.** Consider the product $\Delta \times \Delta^\vee \subset (\mathbb{R}^d)^* \times \mathbb{R}^d$. The complex Σ — our prospective base space — will be a subdivision of

$$|\Sigma| = \{(m, n) \in \Delta \times \Delta^\vee \ : \ \langle m, n \rangle = 1\} \tag{1}$$

which is the union of all sets of the form $F \times F^\vee$, where F runs over all proper faces of Δ, and $F^\vee$ denotes the face (of $\Delta^\vee$) dual to F. The triangulations S and T induce a subdivision of $\Delta \times \Delta^\vee$ into products of simplices, which restricts to a subdivision of $|\Sigma|$.

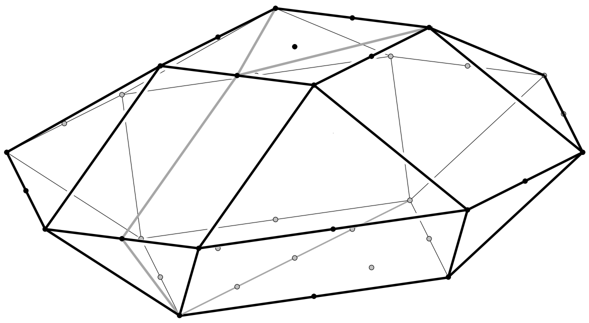


FIGURE 3: The subdivision $S \times T$ of $|\Sigma|$ into products of simplices.

Definition. Σ is the restriction to $|\Sigma|$ of the product subdivision $\operatorname{bsd}(S) \times \operatorname{bsd}(T)$ of $\Delta \times \Delta^\vee$.

The vertices of Σ correspond to pairs $(\widehat{\sigma},\widehat{\tau})$ of (barycenters of) simplices $\sigma \in S$ and $\tau \in T$ with $\langle \sigma, \tau \rangle = 1$. Under this correspondence, the cells of Σ correspond to pairs of chains in the face posets of S and T.

Lemma 2.1. *Σ is isomorphic to the boundary complex of a d-dimensional polytope, and therefore topologically a $(d-1)$-sphere.*

We postpone the proof, and proceed with definitions.

Definition. The singular locus D is the full subcomplex of Σ, induced by vertices $(\widehat{\sigma},\widehat{\tau})$, such that neither σ nor τ is 0-dimensional. (This set indeed induces a subcomplex.)

For $d = 2$, D is empty. For $d = 3$, D will be a finite set of ≤ 24 points (with equality if S and T use all lattice points). For $d = 4$, D will be the first subdivision of a trivalent graph. For all d, the complement $\Sigma \backslash D$ is homotopy equivalent to a bipartite graph.

Definition. Let Γ be the graph on the vertex set $\mathrm{vert}(S) \cup \mathrm{vert}(T)$ with an edge between $v \in \mathrm{vert}(S)$ and $w \in \mathrm{vert}(T)$ if and only if $\langle v, w \rangle = 1$.

Lemma 2.2. *$\Sigma \backslash D$ is homotopy equivalent to Γ.*

First, we introduce some notation that will be used later on. There are two piecewise linear projections

$$p_1 \colon \Sigma \to \mathrm{bsd}(S) \quad \text{and} \quad p_2 \colon \Sigma \to \mathrm{bsd}(T).$$

For a vertex $v \in \mathrm{vert}(S)$ or $w \in \mathrm{vert}(T)$, define U_v respectively V_w to be the preimages

$$U_v = p_1^{-1}(\mathrm{star}_{\mathrm{bsd}(S)}(v)) \quad \text{and} \quad V_w = p_2^{-1}(\mathrm{star}_{\mathrm{bsd}(T)}(w))$$

of open stars in the barycentric subdivisions. Thus, U_v is an open regular neighborhood of the contractible set $v \times (\mathrm{carrier}_\Delta\, v)^\vee$ in Σ, and V_w is an open regular neighborhood of $(\mathrm{carrier}_{\Delta^\vee}\, w)^\vee \times w$. We will abbreviate the collections by $\mathcal{U} = (U_v)_{v \in \mathrm{vert}(S)}$ and $\mathcal{V} = (V_w)_{w \in \mathrm{vert}(T)}$. For future reference, define $\overline{\mathcal{U}} = (\overline{U}_v)_{v \in \mathrm{vert}(S)}$ and $\overline{\mathcal{V}} = (\overline{V}_w)_{w \in \mathrm{vert}(T)}$, as well as $\partial \mathcal{U} = \bigcup \partial U_v$ and $\partial \mathcal{V} = \bigcup \partial V_w$. Observe that with these definitions $\bigcup \mathcal{U} \cup \bigcup \mathcal{V} = \Sigma \backslash D$, and $\partial \mathcal{U} \cap \partial \mathcal{V} = D$.

Proof of Lemma 2.2. Two members U_v and V_w of this covering intersect if and only if

$$v \in (\mathrm{carrier}_{\Delta^\vee}\, w)^\vee \iff w \in (\mathrm{carrier}_\Delta\, v)^\vee \iff \langle v, w \rangle = 1,$$

in which case they intersect in the contractible set $\mathrm{star}_\Sigma((v,w))$. So the claimed homotopy equivalence follows from the nerve lemma (cf. e.g. [Bjö95, Thm. 10.6]). $\square$

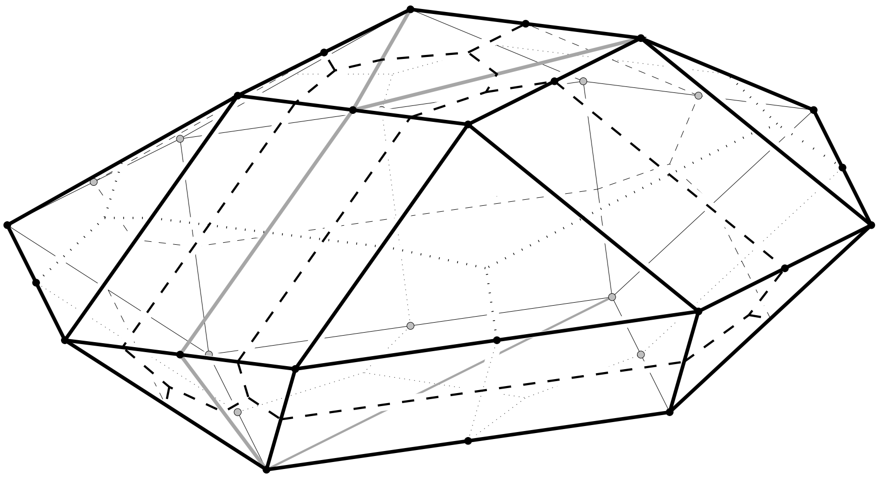


FIGURE 4: The dotted lines are $\partial\mathcal{U}$, and the dashed lines are $\partial\mathcal{V}$. Their intersection D consists of 9 points.

2.2. **The proof of Lemma 2.1.** We will describe a coherent subdivisions of $\Delta_\lambda^\vee$ (alternatively Δ_ν), which is isomorphic to Σ. We hope that the method, which generalizes barycentric subdivisions, may find other applications in the future.

Definition. Suppose that $P \subset Q \subset \mathbb{R}^d$ are polytopes such that P is contained in the relative interior of some face of Q. Then the result of pulling P is by definition the coherent subdivision $\mathrm{pull}_P(Q)$ of Q which is induced by the heights 1 on P, and 0 on all faces of Q that do not contain P.

The case $P \subset \mathrm{relint}(Q)$ is used in [GP88] in order to prove that the space between P and Q can be triangulated without new vertices. We can describe this

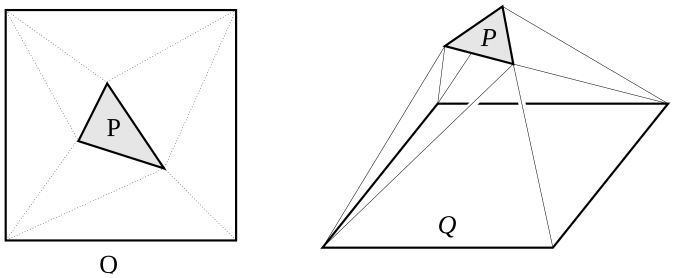


FIGURE 5: The pulling subdivision $\mathrm{pull}_P(Q)$.

subdivision combinatorially as follows. The faces of $\mathrm{pull}_P(Q)$ are all sets of the form $\mathrm{conv}(F \cup F')$ for faces $F \preceq P$ and $F' \prec Q$ with $F \not\subset F'$ (including $F' = \emptyset$) whose normal cones intersect in their relative interiors. That is, there should be a normal vector $n \in (\mathbb{R}^d)^*$ which is maximized over P precisely on F, and over Q precisely on F'. More generally, if P is in the relative interior of a face of a polyhedral complex, then pulling P will affect all faces that contain P in the manner outlined above. If the original subdivision was coherent, then the pulling subdivision will remain coherent. This procedure generalizes the well studied pulling subdivisions where P is a point [Grü67, § 5.2] or [Lee97].

We use these pullings in order to construct a generalized barycentric subdivisions below. We start with a purely combinatorial definition. For a poset $\mathcal{Q}$, the poset/simplicial complex of chains in $\mathcal{Q}$ is denoted $\operatorname{bsd}(\mathcal{Q})$.

Definition. Suppose $\kappa\colon \mathcal{Q} \to \mathcal{P}$ is an order preserving, non-rank-increasing correspondence between the graded posets $\mathcal{Q}$ and $\mathcal{P}$. Define the barycentric subdivision $\operatorname{bsd}(\mathcal{Q}, \kappa)$ of $\mathcal{Q}$ with respect to κ as the subposet

$$\{(p\ ,\ q_0 \prec q_1 \prec \ldots \prec q_r) \in \mathcal{P} \times \operatorname{bsd}(\mathcal{Q})\ :\ p \preceq \kappa(q_0)\}$$

of the product poset $\operatorname{bsd}(\mathcal{Q}) \times \mathcal{P}$.

If $\mathcal{P}$ has only one element, this specializes to $\operatorname{bsd}(\mathcal{Q})$. In our applications, κ will be clear from the context, and will write $\operatorname{bsd}(\mathcal{Q}, \mathcal{P})$ instead.

One example of such a κ arises in the following situation. Say that a polytope $P \subset \mathbb{R}^d$ is a Minkowski summand of the polytope $Q \subset \mathbb{R}^d$, if there is an $\varepsilon > 0$, and a polytope $P' \subset \mathbb{R}^d$ such that $\varepsilon P + P' = Q$. This is true if and only if the normal fan of Q refines the normal fan of P [Smi87]. So there is an order preserving, non-rank-decreasing correspondence on the level of the normal fans which turns into $\kappa\colon \mathcal{L}(Q) \to \mathcal{L}(P)$ on the level of the face lattices.

Definition. Suppose that P is a Minkowski summand of Q. Define the barycentric subdivision $\operatorname{bsd}(Q; P)$ of Q with respect to P as follows. Start with $F = Q$, and proceed by decreasing dimension of faces $F \preceq Q$. Pull the translate $\varepsilon\kappa(F) - \varepsilon\widehat{\kappa(F)} + \widehat{F}$ of the corresponding face of εP in the relative interior of F.

The usual barycentric subdivision appears as the special case where P is a point. Combinatorially, the face lattice $\mathcal{L}(\operatorname{bsd}(Q; P))$ is given by $\operatorname{bsd}(\mathcal{L}(Q), \mathcal{L}(P))$. An

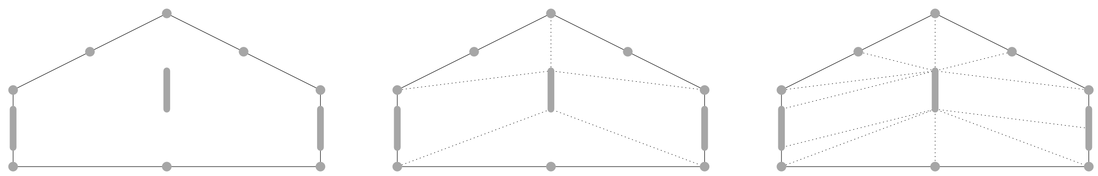


FIGURE 6: The barycentric subdivision of conv $\begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & -2 & 2 & -2 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix} \prec \Delta_\nu$ with respect to the corresponding face conv $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \prec \Delta$.

element $(F_0 \prec F_1 \prec \cdots \prec F_r, G)$ corresponds to the convex hull of the copies of

$G \preceq P$ within the F_i's:

$$\operatorname{conv}\left(\widehat{F_i} + \varepsilon(G - \widehat{\kappa(F_i)})\right) = \operatorname{conv}\left(\widehat{F_i} - \varepsilon\widehat{\kappa(F_i)}\right) + \varepsilon G$$

This polytope is the image under an affine embedding of the product of G with an r-simplex.

The notion generalizes to the situation of two polyhedral complexes with a realized order preserving, non-rank-increasing correspondence between their face posets. We will use this in Section 3.3 for the identity correspondence of $\mathcal{L}(T)$.

Lemma 2.3. *There is a coherent subdivision of $\partial\Delta_\lambda^\vee$ which is combinatorially isomorphic to the restriction to $|\Sigma|$ of the product subdivision* $\operatorname{bsd}(S) \times T$.

Proof. Here, $\Delta^\vee$ is a Minkowski summand of $\Delta_\lambda^\vee$. Let ψ be a piecewise linear concave function with domains of linearity given by $\operatorname{bsd}(\Delta_\lambda^\vee; \Delta^\vee)$. Now ν induces another piecewise linear (non-concave) function $\tilde{\nu}$ on $\Delta_\lambda^\vee$, which is ν in G-direction on $(F_0 \prec F_2 \prec \cdots \prec F_r, G)$, and constant in F_i-direction. The function ψ is strictly

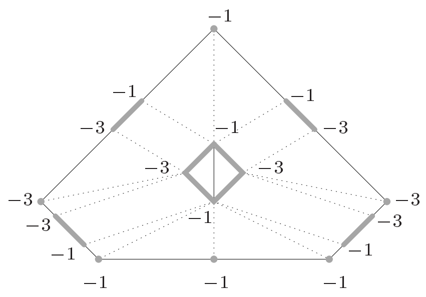


FIGURE 7: The function $\tilde{\nu}$ on $\operatorname{conv}\begin{bmatrix} -2 & 1 & 1 & 2 & 2 \\ 0 & 3 & -3 & 2 & -2 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix} \prec \Delta_\lambda^\vee$.

concave wherever $\tilde{\nu}$ is non-concave, so that for large N, the function $N\psi + \tilde{\nu}$ will be concave. Its domains of linearity are products of simplices that correspond to simplices of $\operatorname{bsd}(\Delta_\lambda^\vee) \cong \operatorname{bsd}(S)$ times simplices of T. $\square$

Now Lemma 2.1 follows if we start from S and $\operatorname{bsd}(T)$ in stead of S and T.

2.3. $\mathrm{SL}(n,\mathbb{Z})$**-structure and monodromy.** An integral affine structure on an n-dimensional manifold Y is given by a torsion-free flat connection on the tangent bundle with holonomy contained in $\mathrm{SL}(n,\mathbb{Z})$. Alternatively, it can be given by a coordinate covering $\{U_\alpha, (y_\alpha^i)_{i=1\ldots n}\}$ of Y, together with transition maps $f_{\alpha\beta} \in \mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{R}^n$ on the non-empty overlaps $U_\alpha \cap U_\beta$ such that the usual cocycle condition is satisfied. This endows the tangent bundle with the well defined lattice structure, generated by the $\{\partial/\partial y_\alpha^i\}$.

Definition. We define an integral affine structure on $\Sigma\backslash D$, using the covering $\mathcal{U} \cup \mathcal{V}$, the nerve of which is the bipartite (two-colored) graph Γ: $U_v \cap V_w$ is non-empty if and only if $\langle v, w\rangle = 1$. For a point $q \in U_v$ we identify the tangent space $T_q(\Sigma\backslash D)$ and the lattice $T_q^{\mathbb{Z}}$ in it with the following codimension 1 subspace and sublattice of the pair $(\mathbb{R}^d, \mathbb{Z}^d)$:

$$T_q = \mathbb{R}^d_v = \{n \in \mathbb{R}^d \ : \ \langle v, n\rangle = 0\}, \qquad T_q^{\mathbb{Z}} = \mathbb{Z}^d_v = \{n \in \mathbb{Z}^d \ : \ \langle v, n\rangle = 0\}.$$

For a point $q \in V_w$ we identify the tangent space $T_q(\Sigma\backslash D)$ and the lattice in it with the $(d-1)$-dimensional quotients

$$T_q = \mathbb{R}^d/w, \qquad T_q^{\mathbb{Z}} = \mathbb{Z}^d/w.$$

On the overlap $U_v \cap V_w$, we define the transition map $f_{vw} : \mathbb{R}^d_v \to \mathbb{R}^d/w$ to be the restriction to the subspace $\mathbb{R}^d_v$ of the natural projection $\mathbb{R}^d \to \mathbb{R}^d/w$.

These transition maps respect the integral structure: $f_{vw} \in \operatorname{Hom}(\mathbb{Z}^d_v, \mathbb{Z}^d/w)$, and the condition $\langle v, w\rangle = 1$ ensures that f_{vw} is an isomorphism. The cocycle condition for the graph-type covering is trivial.

The rest of this subsection is devoted to description of the monodromy. We call a loop in Γ *primary*, if it consists of 4 edges: (v_0, w_0), (w_0, v_1), (v_1, w_1), (w_1, v_0), for some pair of edges $\{v_0, v_1\} \in S$, $\{w_0, w_1\} \in T$. We denote such a loop by $(v_0w_0v_1w_1)$, and think of it as an element of the fundamental group $\pi_1(\Sigma\backslash D) \cong \pi_1(\Gamma)$ with a base point in U_{v_0}.

Lemma 2.4. *The monodromy transformation $T(v_0w_0v_1w_1)\colon \mathbb{Z}^d_{v_0} \to \mathbb{Z}^d_{v_0}$ along the loop $(v_0w_0v_1w_1)$ is given by $T(v_0w_0v_1w_1)(n) = n + \langle v_1, n\rangle(w_1 - w_0)$.*

Proof. Observe that $\langle v_i, w_j\rangle = 1$, $i, j = 0, 1$. Hence, if $n \in \mathbb{Z}^d_{v_0}$, then $n+\langle v_1, n\rangle(w_1-w_0) \in \mathbb{Z}^d_{v_0}$. Now put $n' := n - \langle v_1, n\rangle w_0 \in \mathbb{Z}^d_{v_1}$. Then $n' \equiv n \mod w_0$ together with $n' \equiv n + \langle v_1, n\rangle(w_1 - w_0) \mod w_1$ imply the desired formula. □

The fundamental group $\pi_1(\Sigma\backslash D) \cong \pi_1(\Gamma)$ is a free group whose abelianization $H_1(\Gamma)$ is generated (not freely) by the primary loops. We apply the above calculation to describe the local monodromy near a vertex in the discriminant locus.

Definition. We will call a vertex $(\hat\sigma, \hat\tau) \in D$ to be of *type* (k, l) if $\dim \sigma = k$ and $\dim \tau = l$. Let $\operatorname{star}_\Sigma(\hat\sigma, \hat\tau)$ be its star neighborhood in Σ. Let $\operatorname{vol}(\sigma)$ denote the normalized volume of the simplex σ and same for $\operatorname{vol}(\tau)$.

Theorem 2.5. *The fundamental group $\pi_1(\operatorname{star}_\Sigma(\hat\sigma, \hat\tau)\backslash D)$ is a free group with $k \cdot l$ generators. In a suitable basis the monodromy*

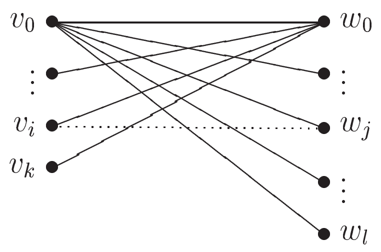
$$T\colon \pi_1(\operatorname{star}_\Sigma(\hat\sigma, \hat\tau)\backslash D) \to \operatorname{SL}(d-1, \mathbb{Z})$$

can be represented (faithfully on the abelianization of π_1*) by an index* $\mathrm{vol}(\sigma)\cdot\mathrm{vol}(\tau)$ *subgroup of the (abelian) group of matrices in the form*

$$\begin{pmatrix} 1 & 0 & \dots & * & \dots & * \\ 0 & 1 & \dots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & * & \dots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

(the identity matrix plus an $l \times k$ *block in the right upper corner).*

Proof. Observe that $\mathrm{star}_\Sigma(\hat\sigma,\hat\tau)\backslash D$ is homotopy equivalent to the subgraph $\Gamma_{(\hat\sigma,\hat\tau)} \subset \Gamma$, a complete bipartite graph with v-nodes labeled by the vertices of σ, and w-nodes labeled by the vertices of τ. A choice of $v_0 \in \sigma$ and $w_0 \in \tau$ determines the spanning tree of $\Gamma_{(\hat\sigma,\hat\tau)}$:



Each edge (v_iw_j), $i = 1,\dots,k$, $j = 1,\dots,l$, added to the spanning tree defines a primary loop which is a generator in the fundamental group of $\Gamma_{(\hat\sigma,\hat\tau)}$. And there are no relations.

Since $\langle v_i, w_j - w_0\rangle = 0$, all $i = 0,\dots,k$, $j = 1,\dots,l$, we can choose a basis $\{e_r\}$ of $\mathbb{Z}^d_{v_0}$ such that

$$w_j - w_0 \in \mathrm{Span}_{\mathbb{Z}}\langle e_1,\dots,e_l\rangle, \quad \text{all } j = 1,\dots,l$$
$$\langle v_i, e_r\rangle = 0, \quad \text{all } i, r = 1,\dots,k.$$

Then, in this basis the monodromy along the primary loop: $(v_0w_0v_iw_j)$

$$T(v_0w_0v_iw_j)(e_r) = e_r + \langle v_i, e_r\rangle(w_j - w_0)$$

will have the desired form.

Finally, the factor of $\mathrm{vol}\,\sigma$ in the index of the subgroup reflects the fact that it may be impossible to complete the collection $\{w_j - w_0\}$ to a full basis of $\mathbb{Z}^d_{v_0}$. Similarly, $\mathrm{vol}\,\tau$ measures the failure of $\{v_i\}$ to form a part of a basis for the dual lattice. $\square$

Remark. If both triangulation T and S are unimodular, then $\mathrm{vol}(\sigma)\cdot\mathrm{vol}(\tau)=1$, and one can choose a basis $\{e_1,\dots,e_{d-1}\}$ of $\mathbb{Z}^d_{v_0}$ such that $e_j=w_j-w_0$ and $\langle v_i,e_r\rangle=\delta_{d-i,r}$. The monodromy matrices then will be

$$T(v_0w_0v_iw_j)=\mathbb{1}+E(j,d-i),$$

where $E(j,d-i)$ is the elementary matrix with 1 in $(j,d-i)$-th place and 0 elsewhere.

2.4. **A glimpse of mirror symmetry.** The invariance of Σ and the discriminant locus D with respect to the duality between triangulated polytopes Δ and $\Delta^\vee$ is manifest. The vertices of D change the type from (k,l) to (l,k). The nodes of the graph Γ also interchange the U- and V-types. And if integral bases associated with vertices of Γ are chosen to be dual to the original ones, then the transition matrices f_{vw} will be replaced by the transpose inverses $(f^t_{vw})^{-1}$. Thus, on the same manifold $Y=\Sigma\backslash D$ the two dual to each other integral affine structures are realized.

Let us introduce notations for the following tori:

$$\mathbb{T}:=\mathbb{R}^d/\mathbb{Z}^d,\qquad \mathbb{T}_v:=(\mathbb{R}^d_v)/(\mathbb{Z}^d_v),\qquad \mathbb{T}/w:=(\mathbb{R}^d/w)/(\mathbb{Z}^d/w).$$

For $\langle v,w\rangle=1$, the transition isomorphism $f_{vw}\in\mathrm{Hom}(\mathbb{Z}^d_v,\mathbb{Z}^d/w)$ induces an isomorphism of the tori, which we will denote by the same symbol

$$f_{vw}\colon \mathbb{T}_v\to\mathbb{T}/w.$$

The torus fibration over $Y=\Sigma\backslash D$ is constructed as follows. Using the affine integral structure on Y one can choose a covariantly constant (with respect to the $\mathrm{SL}_\mathbb{Z}$-connection) lattice $T^\mathbb{Z}Y$ in the tangent bundle TY and form the relative quotient $W\to Y$ with the fibers $W_q=T_qY/T^\mathbb{Z}_qY$. Thus, the fibers are $W_q=\mathbb{T}_v$ when $q\in U_v$, and $W_q=\mathbb{T}/w$ when $q\in V_w$, with the canonical identifications $f_{vw}:\mathbb{T}_v\to\mathbb{T}/w$ for $q\in U_v\cap V_w$.

Let $N(D)\subset\Sigma$ be a regular neighborhood of the discriminant locus. Let $W^\epsilon\to\Sigma\backslash N(D)$ denote the torus fibration associated to the original integral affine structure restricted to the complement of $N(D)$ in Σ. In the next section we will show that the torus fibration W^ϵ on $\Sigma\backslash N(D)$ embeds differentiably into H_s for sufficiently large s. The dual torus fibration by symmetry embeds into the mirror hypersurface. This is the topological part of the mirror symmetry statement in the version of Kontsevich and Soibelman [KS01].

3. Torus fibrations of Calabi-Yau toric hypersurfaces

3.1. **The family.** We describe the standard construction how to obtain the family of Calabi-Yau hypersurfaces from our input data [Bat94]. Recall that $\lambda\in\mathbb{Z}^{\Delta\cap(\mathbb{Z}^d)^*}$ and $\nu\in\mathbb{Z}^{\Delta^\vee\cap\mathbb{Z}^d}$ induce central triangulations $\{0\}*S$ of Δ and $\{0\}*T$ of $\Delta^\vee$. That is, λ and ν lie in the interior of the secondary cone of the respective triangulation [GKZ94].

We use λ to define a (complex) one-parameter family $H_s^{\rm aff}$ of affine hypersurfaces in the complex torus $(\mathbb{C}\backslash\{0\})^d$, and we use ν in order to compactify the torus and the hypersurfaces in a projective toric variety. The affine hypersurfaces are given by

$$H_s^{\rm aff} := \{x \in (\mathbb{C}\backslash\{0\})^d \ : \sum_{m\in\Delta\cap(\mathbb{Z}^d)^*} a_m s^{\lambda(m)} x^m = 0\}$$

The triangulation $\{0\} * T$ of $\Delta^\vee$ induced by the function $\nu : \Delta^\vee \cap \mathbb{Z} \to \mathbb{Z}$ defines a simplicial subdivision of the normal fan to Δ, which in turn defines a projective toric variety X_{Δ_ν} that contains $(\mathbb{C}\backslash\{0\})^d$ as a dense open subset [Ful93, Oda88, Dan78]. Let H_s be the closure of $H_s^{\rm aff}$ in X_{Δ_ν}.

The function ν determines the class of an ample line bundle on X_{Δ_ν}, hence a Kähler class $[\nu] \in H^2(X_{\Delta_\nu}, \mathbb{Z})$. There are several, more or less canonical, ways to define a $\mathbb{T}$-invariant Kähler form on X_{Δ_ν} in the class $[\nu]$. One of the possible constructions of a toric variety is via symplectic reduction. In this case X_{Δ_ν} inherits the natural symplectic structure (which is, in fact, Kähler) from the standard Kähler form $\frac{\sqrt{-1}}{4\pi}\sum dz \wedge d\bar{z}$ on $\mathbb{C}^{\mathrm{vert}(T)}$ (cf. [Gui94]). Alternatively, we can use the pullback of the Fubini-Study form $\omega_{\mathrm{F}S}$ from a projective embedding of X_{Δ_ν}. Though Δ_ν is not necessarily an integral polytope, some k-multiple of it certainly is. We can use the complete linear system given by $k\Delta_\nu$ to define the embedding $i : X_{\Delta_\nu} \hookrightarrow \mathbb{CP}^{(k\Delta_\nu)\cap(\mathbb{Z}^d)^*-1}$. The Kähler form in the class $[\nu] \in H^2(X_{\Delta_\nu}, \mathbb{Z})$ is then given by $\frac{1}{k} i^*\omega_{\mathrm{F}S}$.

In a sense any such "canonical" form ω_0 is unsatisfactory because the metric on H_s defined by restriction of ω_0 to H_s is too far from being Ricci-flat. In the second part of this paper we will describe a family of forms ω_s on X_{Δ_ν}, such that the induced metrics on H_s approximate the Calabi-Yau metrics as $s \to \infty$ much better. (See the Outlook section for more details).

Remark. The toric variety X_{Δ_ν} is simplicial but not necessarily smooth, it may have quotient singularities. Then we can understand the Kähler forms in the orbifold sense (cf., e.g., [AGM93]).

According to [GKZ94, Ch. 10], the hypersurfaces given by equations in the form $\sum b_m x^m = 0$ are all diffeomorphic to each other (in the orbifold sense) as long as the vector $(\log|b_m|)$ $(= \lambda \cdot \log|s| + \log|a|$ in our case) lies in some parallel translation of the $(S * \{0\})$-secondary cone. So that the properties of the family related to the smooth structure do not depend on along which ray we approach the large complex structure point $(s \to \infty)$. Thus, any vector λ in this secondary cone will determine the diffeomorphic torus fibration. For the same reason we can set the coefficients $a_m = 1$ in the defining equation without loss of generality. On the other side, any choice of the Kähler class, as long as it is in the right Kähler cone, also gives rise to the same combinatorics.

The goal of the rest of this section is to exhibit a torus fibration $H_s^{\mathrm{sm}} \to \Sigma\backslash N(D)$ on a "smooth" part of H_s, for large enough s, and show that it is the same as our model fibration $W^\epsilon \to \Sigma\backslash N(D)$.

3.2. **Amoebas of hypersurfaces.** Let $\mathrm{Log}_s : (\mathbb{C}\backslash\{0\})^d \to \mathbb{R}^d$ be the logarithmic map with the base $|s|$:

$$\mathrm{Log}_s(x) := \frac{\log(|x|)}{\log|s|} = \left\{\frac{\log|x_1|}{\log|s|}, \dots, \frac{\log|x_d|}{\log|s|}\right\}.$$

Definition. ([GKZ94, Ch. 6]) The *amoeba* associated to the family of affine hypersurfaces H_s^{aff} is the image of the log map:

$$\mathcal{A}_s^\lambda := \mathrm{Log}_s(H_s^{\mathrm{aff}}).$$

The geometry of amoebas of affine hypersurfaces is a well developed subject that originated in the work of Gelfand, Kapranov and Zelevinsky [GKZ94]. We are going to review several useful facts about the amoebas, most of which are contained in (or can be easily deduced from) a nice survey paper by Mikhalkin [Mik01].

The limiting behavior of amoebas as $s \to \infty$ can be described in terms of the Legendre transform $L_\lambda : \mathbb{R}^d \to \mathbb{R}$ of the vector λ:

$$L_\lambda(n) = \max_{m\in\Delta\cap(\mathbb{Z}^d)^*} \{\langle m, n\rangle + \lambda(m)\}.$$

Remark. In the literature, the Legendre transform is sometimes defined with a "minus" rather than a "plus" sign. Those references work with convex (not concave) λ.

$L_\lambda(n)$ is a piecewise linear convex function. Define the *non-Archimedean amoeba* $\mathcal{A}_\infty^\lambda \subset \mathbb{R}^d$ to be the corner locus of $L_\lambda(n)$ (the set of points where $L_\lambda(n)$ is not smooth). $\mathcal{A}_\infty^\lambda$ is a rational polyhedral complex of dimension $d-1$ (cf. [Mik02]), which gives a cell decomposition of $\mathbb{R}^d$.

Lemma 3.1. *The decomposition of $\mathbb{R}^d$ by $\mathcal{A}_\infty^\lambda$ has the following description:*

1. *The cells are labeled by the simplices $\overline{\sigma} \in S * \{0\}$.*
2. *(The closure of) a cell $Q_{\overline{\sigma}}^\lambda$ is the Minkowski sum of the polytope and the cone:*
   $$Q_{\overline{\sigma}}^\lambda = F_\sigma^\vee + \mathrm{NC}_\Delta(\overline{\sigma}).$$
   Here $F_\sigma^\vee$ is the face of $\partial\Delta_\lambda^\vee$ dual to $\sigma = \overline{\sigma}\cap\partial\Delta \in S$, (where we set $F_\emptyset^\vee = \Delta_\lambda^\vee$ for $\overline{\sigma} = \{0\}$), and $\mathrm{NC}_\Delta(F)$ is the normal cone to the face $F \prec \Delta$, (in particular $\mathrm{NC}_\Delta(F) = \{0\}$ for $F = \Delta$). Thus, $Q_{\overline{\sigma}}^\lambda$ is unbounded if and only if $\overline{\sigma} = \sigma \in S$.
3. *In particular, the d-dimensional cells are labeled by the elements of* $\mathrm{vert}(S)\cup\{0\}$. *That is, there is a bounded central cell $Q_{\{0\}}^\lambda = \Delta_\lambda^\vee$ and unbounded cells Q_v^λ, one for each vertex $v \in \mathrm{vert}(S)$ (see Fig. 8).*

Proof. All statements follow easily from the definition of L_λ. Namely, the cells correspond to the subsets $I \subset \Delta \cap (\mathbb{Z}^d)^*$: the corresponding linear functions $\langle m, n\rangle + \lambda(m)$, $m \in I$, saturate the maximum in L_λ. Since λ is a concave function this can happen only if I is a set of vertices of some simplex $\overline{\sigma} \in S * \{0\}$. This proves (1).

For (3) we notice that a d-cell is a domain of linearity of L_λ, labeled by the vertex $m \in \mathrm{vert}(S) \cup \{0\}$ whose corresponding linear function $\langle m, n\rangle + \lambda(m)$ is maximal. In particular, the central cell $Q^\lambda_{\{0\}}$ is the set of $n \in \mathbb{R}^d$, such that the maximum is achieved by $\langle \{0\}, n\rangle + \lambda(0)$, i.e.

$$\langle m, n\rangle + \lambda(m) \leq \lambda(0), \quad \text{all } m \in \Delta \cap (\mathbb{Z}^d)^*,$$

which are exactly the defining inequalities for $\Delta^\vee_\lambda$.

More generally, a point n is in (the closure of) $Q^\lambda_{\overline{\sigma}}$ if and only if:

$$\begin{aligned}\langle m - v, n\rangle + \lambda(m) \leq \lambda(v) - \lambda(m), &\quad \text{all } v \in \mathrm{vert}\overline{\sigma},\ m \in \Delta \cap (\mathbb{Z}^d)^*,\\ \text{and } \ \langle v_1, n\rangle + \lambda(v_1) = \langle v_2, n\rangle + \lambda(v_2), &\ \ v_1, v_2 \in \mathrm{vert}(\overline{\sigma}),\end{aligned}$$

which are exactly the defining inequalities for the polyhedron $F^\vee_\sigma + \mathrm{NC}_\Delta(\overline{\sigma})$. □

The polyhedral complex $\mathcal{A}^\lambda_\infty$ is also called the *spine* of the amoeba $\mathcal{A}^\lambda_s$ because of the following fact (cf. [Mik02]):

Proposition 3.2. *As $s \to \infty$ the amoebas $\mathcal{A}^\lambda_s$ converge in the Hausdorff sense to the non-Archimedean amoeba $\mathcal{A}^\lambda_\infty$.*

Idea of the proof. If we consider s as a variable, we can think of the affine family H^{aff}_s as one hypersurface given by a single equation in $(\mathbb{C}\backslash\{0\})^{d+1}$. Then the rescaled amoeba $\log|s| \cdot \mathcal{A}^\lambda_s$ sits inside the trace left by this extended $(d+1)$-dimensional amoeba in the horizontal hyperplane $\log|s| = \mathit{const}$. And the result follows from [GKZ94, Ch. 6, Prop. 1.9]. □

Given a $\mathbb{T}$-invariant Kähler form on X_{Δ_ν} in the class $[\nu]$, we can consider the corresponding moment map $\mu\colon X_{\Delta_\nu} \to \Delta_\nu$. In this case we can also define the *compactified amoeba* $\overline{\mathcal{A}^\lambda_s} := \mu(H_s) \subset \Delta_\nu$.

For the proof of Theorem 3.7 we will need to introduce some domains in $\mathbb{R}^d$, which are intimately connected with the amoebas and the function L_λ. In some sense they are generalizations of the cells induced by $\mathcal{A}^\lambda_\infty$.

Definition. For I and J, two disjoint collections of integral points in Δ, and a real number $\epsilon \geq 0$, we define the (possibly empty) polyhedron $Q^\lambda_{(I|J)}(\epsilon)$ in $\mathbb{R}^d$ by the conditions:

$$\langle m'', n\rangle + \lambda(m'') < \langle m, n\rangle + \lambda(m) - \epsilon,$$
$$\langle m', n\rangle + \lambda(m') = \langle m, n\rangle + \lambda(m),$$

for all $m, m' \in I, m'' \notin I \cup J$. We will abbreviate $Q^\lambda_{(I|J)}(\epsilon)$ by $Q^\lambda_I(\epsilon)$ when J is empty.

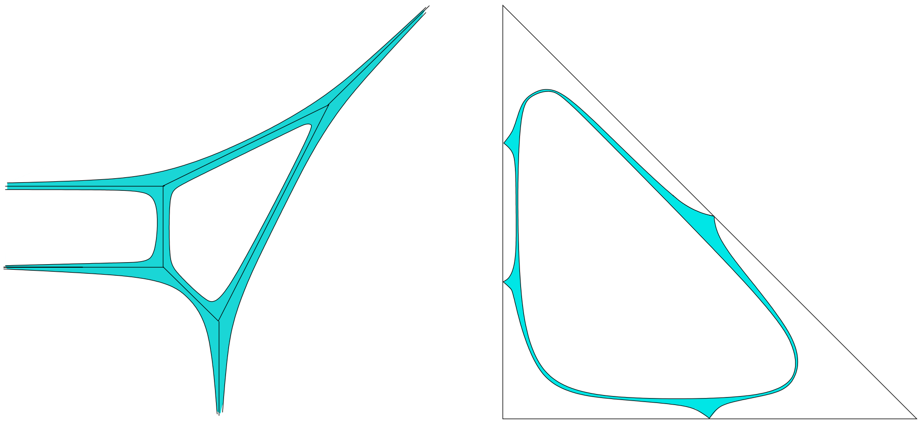


FIGURE 8: The affine amoeba $\mathcal{A}_s^\lambda$ with the corresponding spine $\mathcal{A}_\infty^\lambda$ and its $\mathbb{CP}^2$-compactification $\overline{\mathcal{A}_s^\lambda}$, for the family $H_s^{\mathrm{aff}} = \{s^2 + x^{-1}y^{-1} + sx^{-1} + sx^{-1}y^2 + sx^2y^{-1} = 0\}$.

Definition. For $w \in \partial\Delta^\vee \cap \mathbb{Z}^d$ let $w^\perp$ be the set of integral points in $(\mathrm{carrier}_{\Delta^\vee} w)^\vee$, that is

$$w^\perp = \{m \in \Delta \cap (\mathbb{Z}^d)^* \ : \ \langle m, w\rangle = 1\}.$$

Then, the *truncated* polytope $\Delta \backslash w^\perp$ is defined as the convex hull of integral points of Δ which are not in $w^\perp$.

Notice that for small ϵ, $Q^\lambda_{(I|J)}(\epsilon)$ and $Q^\lambda_{(I|J)}(0)$ are combinatorially equivalent.

Lemma 3.3. *In certain special cases of later interest we can describe $Q^\lambda_{(I|J)}(0)$ as follows:*

(1) *If $J = \emptyset$, then $Q^\lambda_I(0)$ is non-empty if and only if I is the set of vertices of some simplex $\overline{\sigma} \in S * \{0\}$, in which case $Q^\lambda_I(0) = Q^\lambda_{\overline{\sigma}}$.*
(2) *$Q^\lambda_{(\{0\}|v)}(0)$ contains the relative interior of the facet $G_v \subset \Delta^\vee_\lambda$.*
(3) *$Q^\lambda_{(\{0\}|w^\perp)}(0)$ is the Minkowski sum of the polytope $\Delta^\vee_\lambda$ and the normal cone to* $\mathrm{carrier}_{\Delta\backslash w^\perp}\{0\}$*:*

$$Q^\lambda_{(\{0\}|w^\perp)}(0) = \Delta^\vee_\lambda + \mathrm{NC}_{\Delta\backslash w^\perp}(\{0\}).$$

In particular, it contains $\Delta^\vee_\lambda + \mathrm{cone}(w)$.

Proof. For (1) we notice that the set of defining inequalities of $Q^\lambda_I(0)$ is exactly the condition that the maximum in L_λ is saturated by the linear functions $\langle m, n\rangle + \lambda(m)$, $m \in I$.

For (2), note that $Q^\lambda_{v*\{0\}}$ is defined by the same inequalities as $Q^\lambda_{(\{0\}|v)}(0)$, plus an extra condition: $\langle v, n\rangle + \lambda(v) = \lambda(0)$.

For (3) we can study the Legendre transform of the restriction of λ to the truncated polytope $\Delta\backslash w^\perp$ (which is still a concave function). From this point of view, the Minkowski sum in (3) is in complete analogy with (2) of Lemma 3.1.

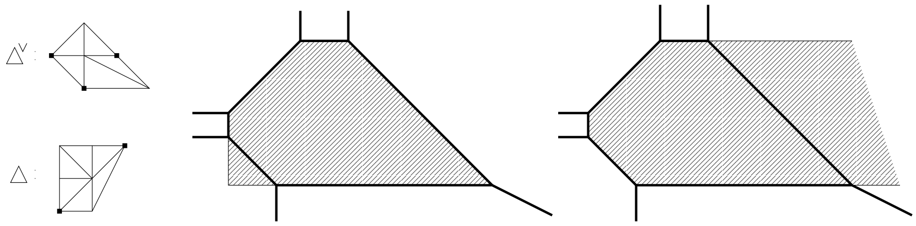


FIGURE 9: Examples of $Q^{\lambda}_{(\{0\}|v)}(0)$ for $v = \left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ in $\partial\Delta$.

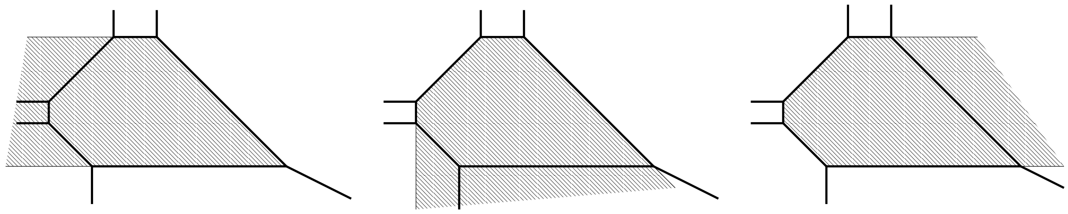


FIGURE 10: Examples of $Q^{\lambda}_{(\{0\}|w^{\perp})}(0)$ for $w = \left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ in $\partial\Delta^{\vee}_{\lambda}$.

Note that in the process of truncation we removed all integral points of Δ with $\langle m, w\rangle = 1$. Hence, the remaining ones satisfy $\langle m, w\rangle \leq 0$ which implies that $\{0\}$ is on the boundary of $\Delta\backslash w^{\perp}$, and w is in (the boundary of) the normal cone $\mathrm{NC}_{\Delta\backslash w^{\perp}}(\{0\})$. $\square$

3.3. **Foliation of $\mathbb{R}^d\backslash Q^{\lambda}_{\{0\}}(\epsilon)$.** We will exhibit a vector field on $\mathbb{R}^d\backslash Q^{\lambda}_{\{0\}}(\epsilon)$ with values in $\partial\Delta^{\vee}_{\lambda} \subset \mathbb{R}^d$. Its integral curves yield the desired foliation $\mathcal{F}$. Recall from Lemma 2.3 that there is a subdivision of $\partial\Delta^{\vee}_{\lambda}$ which is combinatorially isomorphic to $\mathrm{bsd}(S) \times T$, restricted to $|\Sigma|$. In this context, the projection p_2 can be thought of as defined $\partial\Delta^{\vee}_{\lambda} \to \partial\Delta^{\vee}$, and can be interpreted as a vector field on $\partial\Delta^{\vee}_{\lambda}$. We will deform p_2, and extend the deformed vector field to $\mathbb{R}^d\backslash Q^{\lambda}_{\{0\}}(\epsilon)$.

For the deformation part, we use that the faces of $\mathrm{bsd}(T, T)$ are parameterized by pairs $(\tau,\, \mathfrak{x} = \tau_0 \prec \ldots \prec \tau_r) \in T \times \mathrm{bsd}(T)$ with $\tau \preceq \tau_0$ (cf. § 2.2).

Definition. In the δ-realization of $\mathrm{bsd}(T, T)$, the face of $\mathrm{bsd}(T, T)$ which corresponds to $(\tau,\mathfrak{x})$ is the Minkowski sum $\delta\tau + (1 - \delta)\mathfrak{x}$.

The δ-realization of $\mathrm{bsd}(T, T)$ yields a cellular map $\mathfrak{d}^{\delta}\colon\ \mathrm{bsd}(T, T) \to T$, which maps $(\tau,\, \mathfrak{x})$ to τ. (I.e., the small copy of τ in itself is stretched to full size, and the space between the small copies is collapsed into the smallest face.

Lemma 3.4. *If $\tau_1 \prec \tau_2$ are simplices of T, then $\mathfrak{d}^{\delta}$ is invariant with respect to $\widehat{\tau}_1 - \widehat{\tau}_2$ on the region $\bigcup_{\theta\in[0,1-\delta]} \theta\widehat{\tau}_2 + (1 - \theta)\tau_1$. In particular, $\mathfrak{d}^{\delta}$ is constant equal to w on the whole star of $w \in \mathrm{vert}(T)$ in $\mathrm{bsd}(T, T)$.*

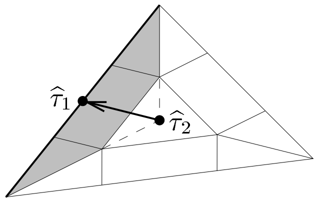


FIGURE 11: The map $\mathfrak{d}^\delta$ is $(\widehat{\tau}_1 - \widehat{\tau}_2)$-invariant in the shaded region.

Lemma 3.5. *Let K be a subcomplex of* $\operatorname{bsd}(T)$*, and let N be a neighborhood of K. Then there is a δ-realization of* $\operatorname{bsd}(T,T)$ *such that for each face $\mathfrak{x}$ of K, the faces of* $\operatorname{bsd}(T,T)$ *which correspond to some $(\tau,\, \mathfrak{x})$ are contained in N.*

We will apply Lemma 3.5 for $K = p_2(\partial\mathcal{V})$, and $N = p_2(N_2(\partial\mathcal{V}))$, where $N_2(\partial\mathcal{V})$ is a neighborhood of $\partial\mathcal{V}$ in Σ.

Proof. Choose δ small enough to ensure that $\mathfrak{x} + \delta(\tau_0 - \widehat{\tau}_0) \subset N$ for every simplex $\mathfrak{x} = (\tau_0 \prec \ldots \prec \tau_r) \in K$.

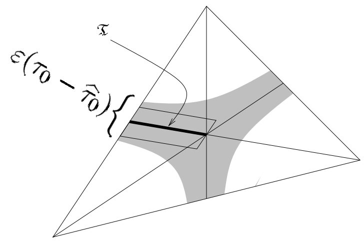


FIGURE 12: Choice of δ in the proof of Lemma 3.5.

Then the maximal cell which corresponds to $(\tau_0,\, \mathfrak{x})$ is given by

$$\delta\tau_0 + (1-\delta)\mathfrak{x} \subseteq \mathfrak{x} + \delta(\tau_0 - \widehat{\tau}_0) \subset N.$$

□

Now we are ready to define the vector field $\mathfrak{X}^\delta$ on $\partial\Delta_\lambda^\vee$ as the composition $\mathfrak{d}^\delta p_2 \colon \partial\Delta_\lambda^\vee \to \partial\Delta^\vee$. In order to extend $\mathfrak{X}^\delta$ to both sides of $\partial\Delta_\lambda^\vee$, we present a polyhedral subdivision of a neighborhood of $\partial\Delta_\lambda^\vee$ whose trace on $\partial\Delta_\lambda^\vee$ realizes the restriction of $\operatorname{bsd}(S) \times T$ to $|\Sigma|$.

Remark. In § 2.2, we were merely interested in the sphericity of $|\Sigma|$. We left open where to place the small copies of faces, and how small we wanted these copies to be. In the following we fix a realization of this subdivision which we will keep through the remainder of the article. In particular, ϵ is a fixed constant.

Denote by $\lambda^\epsilon \in \mathbb{R}^{\Delta\cap(\mathbb{Z}^d)^*}$ the vector given by $\lambda^\epsilon(0) = \lambda(0)$, and $\lambda^\epsilon(v) = \lambda(v) + \epsilon$ for $v \in \mathrm{vert}(S)$. Suppose that $\epsilon > 0$ is small enough to ensure that λ and λ^ϵ induce the same triangulation. Then $\Delta^\vee_{\lambda^\epsilon} \subset \Delta^\vee_\lambda$ are combinatorially equivalent. For a simplex $\mathfrak{s} \in \mathrm{bsd}(S)$, denote $\mathfrak{s}_\epsilon$ the corresponding simplex of $\mathrm{bsd}(\Delta^\vee_{\lambda^\epsilon})$. (I.e., $\mathfrak{s} \subset (\mathbb{R}^d)^*$, while $\mathfrak{s}_\epsilon \subset \mathbb{R}^d$.) Given a simplex $\tau \in T$ with $\langle \mathfrak{s}, \tau\rangle = 1$, we can form the Minkowski sum $\mathfrak{s}_\epsilon + \mathbb{R}_{\geq\epsilon/2}\tau$. These fit together to form a complex of (unbounded) polyhedra which subdivides $\mathbb{R}^d$ outside $\Delta^\vee_{\lambda^{\epsilon/2}}$. (It actually refines the subdivision into $Q^{\lambda^{\epsilon/2}}_\sigma$'s provided by the non-Archimedean amoeba for $\lambda^{\epsilon/2}$.)

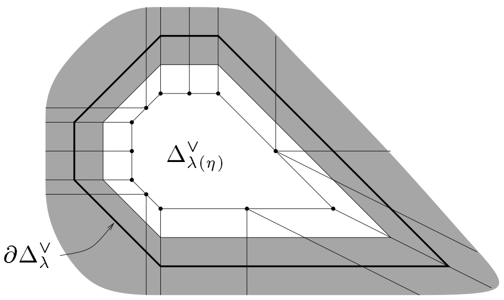


FIGURE 13: The polyhedral subdivision of $\mathbb{R}^d$ outside $\Delta^\vee_{\lambda^{\epsilon/2}}$.

Definition. For $0 < \delta < 1/2$, the vector field $\mathfrak{X}^\delta\colon \mathbb{R}^d\backslash\Delta^\vee_{\lambda^{\epsilon/2}} \to \partial\Delta^\vee \subset \mathbb{R}^d$ is the unique vector field which agrees with $\mathfrak{d}^\delta p_2$ on $\partial\Delta^\vee_\lambda$, and is invariant with respect to $\widehat{\tau}$ on the polyhedron $\mathfrak{s}_\epsilon + \mathbb{R}_{\geq\epsilon/2}\tau$.

We need to argue that this determines a continuous vector field. The problem may arise only when we try to assign a vector to a point $n \in \mathfrak{s}_\epsilon + \mathbb{R}_{\geq\epsilon/2}\tau$ such that there are two points n_1, n_2 which already have a vector assigned to them, so that both $n - n_1$ and $n - n_2$ are a multiple of $\widehat{\tau}$. Then $n_1 \in \mathfrak{s}_\epsilon + [\epsilon/2, \epsilon]\tau_1$ for some face $\tau_1 \prec \tau$, and $\mathfrak{X}^\delta(n_1)$ is defined as $\mathfrak{X}^\delta(n'_1)$, where $n'_1 \in \partial\Delta^\vee_\lambda$, and $n_1 - n'_1$ is a multiple of $\widehat{\tau}_1$. Furthermore, $n_2 \in \partial\Delta^\vee_\lambda$, and $n_2 - n'_1$ is a multiple of $\widehat{\tau} - \widehat{\tau}_1$. Here we use the assumption that $\delta < 1/2$ to conclude that $\mathfrak{X}^\delta(n_2) = \mathfrak{d}^\delta p_2(n_2) = \mathfrak{d}^\delta p_2(n'_1) = \mathfrak{X}^\delta(n'_1)$.

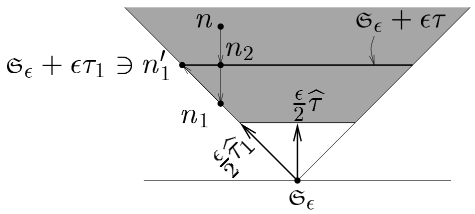


FIGURE 14: $\mathfrak{X}^\delta(n)$ is doubly defined: via $\mathfrak{X}^\delta(n_1) = \mathfrak{X}^\delta(n'_1)$, and via $\mathfrak{X}^\delta(n_2)$.

The integral curves of $\mathfrak{X}^\delta$ foliate $\mathbb{R}^d \backslash \Delta^\vee_{\lambda^{\epsilon/2}}$. The following lemma summarizes the main properties of $\mathfrak{X}^\delta$ and the foliation $\mathcal{F}$.

Lemma 3.6. *Given a neighborhood $N_2(\partial\mathcal{V})$ of $\partial\mathcal{V} \subset \partial\Delta^\vee_\lambda \cong \Sigma$, there is a $\delta > 0$ such that*

(1) *If $n \in Q_v^{\lambda^\epsilon}$, then $\langle v, \mathfrak{X}^\delta(n)\rangle = 1$.*
(2) *If $n \in V_w \backslash N_2(\partial\mathcal{V})$, the flow line $\mathcal{F}_n$ through n is a straight line parallel to w outside $\Delta^\vee_{\lambda^\delta}$.*

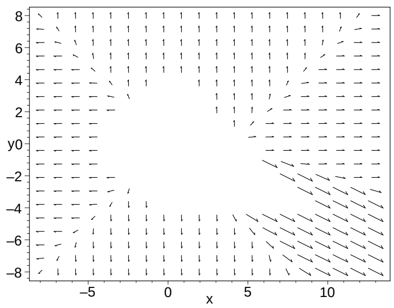


FIGURE 15: The vector field $\mathfrak{X}^\delta$ for a 2-dimensional example.

Proof. Choose $0 < \delta < 1/2$ so that the 2δ-realization of $\mathrm{bsd}(T,T)$ satisfies the conclusion of Lemma 3.5 for $K = \partial\mathcal{V}$ and $N = p_2(N_2(\partial\mathcal{V}))$.

By construction of $\mathfrak{X}^\delta$, the set of values on one of the polyhedra $\mathfrak{S}_\epsilon + \mathbb{R}_{\geq \epsilon/2}\tau$ is contained in the set of values on its boundary which is contained in τ. Statement (1) follows from $\langle \mathfrak{S}, \tau\rangle = 1$.

For (2), let $n \in V_w \backslash N_2(\partial\mathcal{V})$. Then $p_2(n)$ belongs to a cell $(\mathfrak{T}, w)$ of the 2δ-realization of $\mathrm{bsd}(T,T)$, so that $\mathfrak{X}^\delta(n) = w$. Also, say, $n \in \overline{U}_v$. If we parameterize $\mathcal{F}_n(t)$ such that $\mathcal{F}_n(0) = n$ (and $\dot{\mathcal{F}}_n(t) = \mathfrak{X}(\mathcal{F}_n(t))$), then, by (1), $\langle v, \mathcal{F}_n(t)\rangle = \lambda(v) + t$. So $\mathcal{F}_n(t) - t\widehat{\tau}$ stays in the hyperplane $\langle v, \cdot\rangle = 1$, where $\tau = \mathrm{carrier}_T(\mathfrak{T})$.

For $t > 0$, $\mathcal{F}_n(t) = n + tw$. For $-\delta < t < 0$, let $\ell \in (\mathbb{R}^d)^*$ be a linear functional which takes the values 0 on w, and 1 on the opposite side of $\mathfrak{T}$. Then $\ell(p_2(n)) < 1 - 2\delta$, and $\frac{d}{dt}\ell(p_2(\mathcal{F}_n(t) - t\widehat{\tau})) = \ell(w - \widehat{\tau}) = 1$. So in this time range, $\mathcal{F}_n(t) = n + tw$ as well. For $t < -\delta$, $\mathcal{F}_n(t)$ belongs to $\Delta^\vee_{\lambda^\delta}$ by (1). □

Remark. The foliation $\mathcal{F}$ can be, in fact, continued to the boundary of Δ_ν (not smoothly at the $d-2$-skeleton of Δ_ν) via the diffeomorphism $\mu \circ \mathrm{Log}_s^{-1}$ between $\mathbb{R}^d$ and the interior of Δ_ν. So that it will induce a projection $X_{\Delta_\nu} \backslash \mathrm{Log}_s^{-1}(Q^\lambda_{\{0\}}(\epsilon)) \to \Sigma$. But to construct torus fibrations we will use only a part of this projection where it is clearly well defined. That is why we do not provide a proof for this more general statement here.

3.4. **The torus fibration.** Using the foliation $\mathcal{F}$ we are going to define a decomposition of the hypersurface $H_s = H_s^{\mathrm{sm}} \sqcup H_s^{\mathrm{sing}}$, construct a torus fibration $H_s^{\mathrm{sm}} \to \Sigma\backslash N(D)$ and show that it is isomorphic to the fibration $W^\epsilon \to \Sigma\backslash N(D)$.

For any closed subset $J \subset \Sigma$ we will denote by $X_s(J) \subset X_{\Delta_\nu}$ the closure of $\mathrm{Log}_s^{-1}\left(\bigcup_{q\in J} \mathcal{F}_q\right)$ in X_{Δ_ν}.

Definition. Let $N(D)$ be a regular neighborhood of D in Σ. Then the *smooth* part of the hypersurface is $H_s^{\mathrm{sm}} := H_s \cap X_s(\Sigma\backslash N(D))$, and the rest $H_s^{\mathrm{sing}} := H_s\backslash H_s^{\mathrm{sm}}$ is *singular.*

Since $D = \partial\mathcal{U}\cap\partial\mathcal{V}$, there exist regular neighborhoods $N_1(\partial\mathcal{U})$ of $\partial\mathcal{U}$ and $N_2(\partial\mathcal{V})$ of $\partial\mathcal{V}$ in Σ, such that $N(D) \supset N_1(\partial\mathcal{U}) \cap N_2(\partial\mathcal{V})$. This means that $\Sigma\backslash N(D)$ can be covered by the union of the closed sets:

$$\mathcal{U}^\epsilon = \{U_v^\epsilon\} = \{U_v\backslash N_1(\partial\mathcal{U})\} \quad \text{and} \quad \mathcal{V}^\delta = \{V_w^\delta\} = \{V_w\backslash N_2(\partial\mathcal{V})\}.$$

The amoebas $\mathcal{A}_s^\lambda$, for a large enough s, all lie in $\mathbb{R}^d\backslash Q_{\{0\}}^\lambda(\epsilon)$. This means that $\mathcal{F}$ defines a projection $\mathcal{A}_s^\lambda \to \Sigma$ and, by composition with Log_s, the projection $H_s^{\mathrm{aff}} \to \Sigma$. Also $\mathcal{A}_s^\lambda$ lie in $\mathbb{R}^d\backslash Q_v^\lambda(\epsilon)$, for any $v \in \mathrm{vert}(S)$ and large s. Since the unbounded ends of flow lines $\mathcal{F}_q$, for $q \in U_v^\epsilon$, are in $Q_v^\lambda(\epsilon)$ their closures do not contain any extra points of the hypersurface:

$$H_s^{\mathrm{aff}} \cap \mathrm{Log}_s^{-1}\left(\bigcup_{q\in U_v^\epsilon} \mathcal{F}_q\right) = H_s^{\mathrm{aff}} \cap X_s(U_v^\epsilon) = H_s \cap X_s(U_v^\epsilon).$$

Thus, the map $H_s \cap X_s(U_v^\epsilon) \to U_v^\epsilon$ is well defined. On the other hand, for two distinct points q_1, q_2 in V_w^δ the corresponding leaves are straight lines. Written in local coordinates (see Lemma 3.9) this implies that the sets $X_s(\mathcal{F}_{q_1})$ and $X_s(\mathcal{F}_{q_2})$ are disjoint. Hence, the map $H_s\cap X_s(V_w^\delta) \to V_w^\delta$ is well defined. Combined together we have (for large enough s) the well defined projection

$$f_s\colon H_s^{\mathrm{sm}} \to \Sigma\backslash N(D), \qquad f_s(x) := q \;\Leftrightarrow\; x \in X_s(q).$$

Theorem 3.7. *There exists a real number s_0, such that for any s with $|s| \geq s_0$,*

$$f_s\colon H_s^{\mathrm{sm}} \to \Sigma\backslash N(D)$$

is a torus fibration isomorphic to $W^\epsilon \to \Sigma\backslash N(D)$.

Before proving the theorem we need to make a comment about smoothness. H_s^{sm} is missing all singular points (if any) of the toric variety X_{Δ_ν}, which are all in the moment map preimage of the $(d-2)$-skeleton of Δ_ν (see Lemma 3.9). On the other hand, $\Sigma\backslash N(D)$ carries a canonical smooth structure induced by the affine structure on $\Sigma\backslash D$. So given the topological fibration $H_s^{\mathrm{sm}} \to \Sigma\backslash N(D)$ of Theorem 3.7, standard techniques apply to make it smooth.

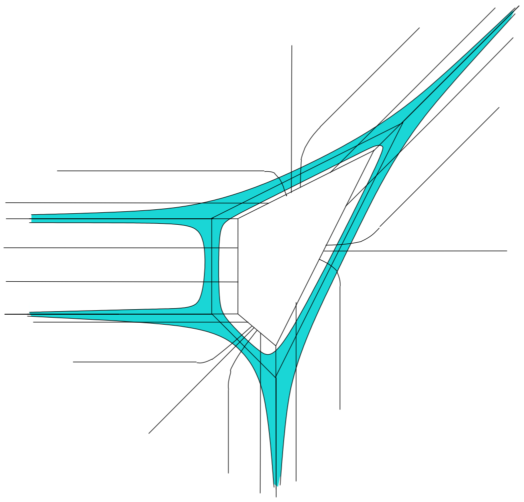


FIGURE 16: The foliation $\mathcal{F}$ of $\mathbb{R}^d \backslash Q^{\lambda}_{\{0\}}(\epsilon)$ induces a foliation of the amoeba $\mathcal{A}^{\lambda}_s$.

The strategy of proving Theorem 3.7 will be as follows. First, we analyze the map f_s in U^{ϵ}_v and V^{δ}_w for every $v \in \mathrm{vert}(S), w \in \mathrm{vert}(T)$. Then the proof of the theorem can be completed in three steps: we show that $f_s \colon H^{\mathrm{sm}}_s \to \Sigma \backslash N(D)$ is a torus fibration over the two kinds of covering patches, and then check that it has the monodromy of our model.

Let $v \in \mathrm{vert}(S)$. For a fixed s we consider the $(\Delta \cap (\mathbb{Z}^d)^* - 2)$-parameter family of hypersurfaces $H^v_s(a)$ in $X_s(U^{\epsilon}_v)$:

$$s^{\lambda(0)} + s^{\lambda(v)} x^v + \sum_{m \neq \{0\}, v} a_m s^{\lambda(m)} x^m = 0, \quad 0 \le a_m \le 1.$$

Lemma 3.8. *There exists s_0 such that whenever $|s| \ge s_0$, all $H^v_s(a)$ are smooth and transversal to $X_s(q)$ for every $q \in U^{\epsilon}_v$.*

Proof. According to Proposition 3.2 we can choose s big enough so that the Log_s-image of every hypersurface $H^v_s(a)$ lies in the ϵ-neighborhood of U^{ϵ}_v. Recall from Lemma 3.3 that a small neighborhood of U^{ϵ}_v lies in the domain $Q^{\lambda}_{(\{0\}|v)}(\epsilon)$. Thus we can assume that all hypersurfaces $H^v_s(a)$ lie entirely in $\mathrm{Log}_s^{-1}(Q^{\lambda}_{(\{0\}|v)}(\epsilon))$.

Whenever $\mathrm{Log}_s(x) \in Q^{\lambda}_{(\{0\}|v)}(\epsilon)$, we have

$$\langle m, \frac{\log |x|}{\log |s|} \rangle + \lambda(m) \le \lambda(0) - \epsilon, \text{ for all } m \neq v, \{0\}$$

or, equivalently,

$$|x^m s^{\lambda}(m)| \le |s|^{-\epsilon} |s|^{\lambda(0)}.$$

This means that the values of all monomials $x^m s^{\lambda(m)}$, $m \neq v, \{0\}$, for $x \in \mathrm{Log}_s^{-1}(Q^{\lambda}_{(\{0\}|v)}(\epsilon))$, are (uniformly) bounded by $|s|^{-\epsilon}|s|^{\lambda(0)}$. Note also, that their log-derivatives are bounded by $C|s|^{-\epsilon}|s|^{\lambda(0)}$, some constant $C \geq 0$, since

$$x\frac{\partial}{\partial x}(a_m s^{\lambda(m)} x^m) = m \cdot a_m s^{\lambda(m)} x^m.$$

For any basis $\{e_i\}$ of $(\mathbb{Z}^d)^*$, the functions $y_i = x^{e_i}$ give affine coordinates on $(\mathbb{C}\backslash\{0\})^d$. We choose $e_1 = -v$, multiply the equations of the hypersurfaces in our family $H_s^v(a)$ by $y_1 = x^{-v}$, and look for critical points:

$$\begin{aligned}
&\frac{\partial}{\partial y_1}\big(y_1 s^{\lambda(0)} + s^{\lambda(v)} + y_1 \sum_{m \neq \{0\},v} a_m s^{\lambda(m)} x^m\big) \\
&\qquad = s^{\lambda(0)} + \big(1 + y_1\frac{\partial}{\partial y_1}\big) \sum_{m \neq \{0\},v} a_m s^{\lambda(m)} x^m = s^{\lambda(0)}(1 + O(|s|^{-\epsilon})) \neq 0,
\end{aligned}$$

for large enough s. Thus, there are no critical points, hence every member of our family $H_s^v(a)$ is smooth.

Finally, note that $\bigcup_{q \in U_v^\epsilon} \mathcal{F}_q$ is in $Q_v^{\lambda^\epsilon}$, but Lemma 3.6 asserts that the vectors $\xi \in \mathfrak{X}$ in $Q_v^{\lambda^\epsilon}$ satisfy $\langle v, \xi \rangle = 1$. Thus, for any point of intersection $H_s^v(a) \cap X_s(q)$ the corresponding tangent vector to $X_s(q)$ has the form:

$$\bar{\xi} = y_1\frac{\partial}{\partial y_1} + \alpha_2 y_2 \frac{\partial}{\partial y_2} + \cdots + \alpha_d y_d \frac{\partial}{\partial y_d}.$$

Differentiating the defining equation for $H_s^v(a)$ with respect to $\bar{\xi}$ gives:

$$\begin{aligned}
&\bar{\xi}\big(y_1 s^{\lambda(0)} + s^{\lambda(v)} + y_1 \sum_{m \neq \{0\},v} a_m s^{\lambda(m)} x^m\big) \\
&\qquad = y_1 s^{\lambda(0)}(1 + O(|s|^{-\epsilon})) + \sum_{i=2}^{d} \alpha_i y_i \frac{\partial}{\partial y_i}\big(y_1 \sum_{m \neq \{0\},v} a_m s^{\lambda(m)} x^m\big) \\
&\qquad = y_1 s^{\lambda(0)}(1 + O(|s|^{-\epsilon})) + \sum_{i=2}^{d} y_1 s^{\lambda(0)} O(|s|^{-\epsilon}) = y_1 s^{\lambda(0)}(1 + O(|s|^{-\epsilon})) \neq 0.
\end{aligned}$$

Thus, we can conclude that $\bar{\xi}$ is transversal to the tangent planes to $H_s^v(a)$, that is $X_s(q)$ is transversal to all $H_s^v(a)$. □

Remark. The estimates for the monomials in the lemma can be used to give another proof of the Hausdorff convergence in Proposition 3.2. Note that for any $\epsilon > 0$ for large enough s, the monomial $x^v s^{\lambda(v)}$ become dominant in $\mathrm{Log}_s^{-1}(Q_v^\lambda(\epsilon))$. Hence the equation for H_s^{aff} cannot have solutions in this domain. This means that the amoebas $\mathcal{A}_s^\lambda$ are ϵ-close to their spine $\mathcal{A}_\infty^\lambda$.

Now let $w \in \mathrm{vert}(T)$. Recall that $w^\perp$ is the set of integral points in $(\mathrm{carrier}_{\Delta^\vee} w)^\vee$. For a fixed s we consider the $(\Delta \cap (\mathbb{Z}^d)^* - w^\perp - 1)$-parameter family of hypersurfaces:

$$s^{\lambda(0)} + \sum_{m \in G_w} s^{\lambda(m)} x^m + \sum_{m \notin G_w \cup \{0\}} a_m s^{\lambda(m)} x^m = 0, \quad 0 \le a_m \le 1,$$

and let $H_s^w(a)$ be its closure in $X_s(V_w^\delta)$. Now we can repeat the arguments of Lemma 3.8 to prove the analogous statement for the family $H_s^w(a)$.

Lemma 3.9. *There exists s_0 such that whenever $|s| \ge s_0$, all $H_s^w(a)$ are smooth and transversal to $X_s(q)$ for every $q \in V_w^\delta$.*

Proof. According to Proposition 3.2 we can choose s big enough so that the Log_s-image of the affine part of every hypersurface $H_s^w(a)$ lies in the ϵ-neighborhood of the Minkowski sum $V_w^\delta + \mathrm{cone}(w)$. Also, recall from Lemma 3.3 that $V_w^\delta + \mathrm{cone}(w)$ lies in the domain $Q^\lambda_{(\{0\}|w^\perp)}(\epsilon)$. Thus, we can assume that the affine parts of all hypersurfaces $H_s^w(a)$ lie in $\mathrm{Log}_s^{-1}(Q^\lambda_{(\{0\}|w^\perp)}(\epsilon))$.

We choose a basis $\{e_i\}$ of $(\mathbb{Z}^d)^*$ such that

$$\langle e_1, w\rangle = -1 \text{ and } \langle e_i, w\rangle = 0, i = 2, \dots, d.$$

Then the affine coordinate functions $y_i = x^{e_i}$ can be extended (by allowing zero values for y_1) to the open part of the toric divisor Z_w corresponding to the facet $F_w \subset \partial\Delta_\nu$. Moreover, in these coordinates the preimage of each flow line $\mathcal{F}_q$ in $(\mathbb{C}\backslash\{0\})^d$ is defined by fixing the values of $|y_2|, \dots, |y_d|$, so that its closure $X_s(q)$ is defined by the same equations, but allowing the zero value for y_1. Hence, we can use $\{y_i\}$ as global coordinates on $X_s(V_w^\delta)$.

Multiplying the affine equation of $H_s^w(a)$ by y_1 we note that the Laurent polynomial

$$y_1 \sum_{m \in G_w} s^{\lambda(m)} x^m + y_1 \sum_{m \notin G_w \cup \{0\}} a_m s^{\lambda(m)} x^m$$

has only positive powers of y_1, where as its first part $P_1(y) = y_1 \sum_{m \in G_w} s^{\lambda(m)} x^m$ is independent of y_1 at all. Thus, we get the global equation for the family in $X_s(V_w^\delta)$.

Now we can repeat the estimates for the monomials and their log-derivatives. Whenever $\mathrm{Log}_s(x) \in Q^\lambda_{(\{0\}|w^\perp)}(\epsilon)$, we have

$$\langle m, \frac{\log|x|}{\log|s|}\rangle + \lambda(m) \le \lambda(0) - \epsilon, \text{ for all } m \notin G_w \cup \{0\},$$

or, equivalently,

$$|x^m s^\lambda(m)| \le |s|^{-\epsilon} |s|^{\lambda(0)}.$$

When written in the y-coordinates these estimates extends by continuity from the affine part to the entire $X_s(V_w^\delta)$.

To see that $H_s^w(a)$ has no critical points we differentiate its defining equation with respect to y_1:

$$\frac{\partial}{\partial y_1}\big(y_1 s^{\lambda(0)} + y_1 \sum_{m\in G_w} s^{\lambda(m)}x^m + y_1 \sum_{m\notin G_w\cup\{0\}} a_m s^{\lambda(m)}x^m\big)$$
$$= s^{\lambda(0)} + \big(1 + y_1\frac{\partial}{\partial y_1}\big)\sum_{m\notin G_w\cup\{0\}} a_m s^{\lambda(m)}x^m = s^{\lambda(0)}(1+O(|s|^{-\epsilon}))\neq 0,$$

for large enough s.

Finally, Lemma 3.6 asserts that the vector field $\mathfrak{X}$ in $\bigcup_{q\in V_w^\delta}\mathcal{F}_q$ is constant and equal to w. It means that $\frac{\partial}{\partial y_1}$ is a tangent vector to $X_s(q)$, $q\in V_w^\delta$, and it is transversal to $H_s^w(a)$ by the above calculation. □

Proof of Theorem 3.7. Note that if all $a_i=1$ in family $H_s^v(a)$, then we have the original equation of H_s. On the other hand, if all $a_i=0$, then the family $H_s^v(a)$ degenerates to the hyperbola:

$$H_s^v(0) := \{x\in X_s(U_v^\epsilon)\ :\ s^{\lambda(0)} + s^{\lambda(v)}x^v = 0\}.$$

Because $X_s(\mathcal{F}_q)$, $q\in U_v^\epsilon$, intersect every $H_s^v(a)$ transversally, the corresponding fibers $F_q := H_s\cap X_s(\mathcal{F}_q)$ and $F_q^v := H_s^v(0)\cap X_s(\mathcal{F}_q)$ are diffeomorphic.

If $\theta=\{\theta_i\}$ denote the coordinates of the torus $\mathbb{T}$ and θ_s is the phase of s, then the fiber F_q^v of $H_s^v(0)$ is the torus

$$F_q^v = \{\theta\in\mathbb{T}\ :\ \langle v,\theta\rangle + (\lambda(0)-\lambda(v))\theta_s \equiv 0 \bmod 2\pi\},$$

which, for a fixed s, can be identified with the torus $\mathbb{T}_v$ (though, see the remark below about monodromy as $\theta_s\mapsto\theta_s+2\pi$).

Similarly, the fibers $F_q = H_s\cap X_s(\mathcal{F}_q)$ and $F_q^w := H_s^w(0)\cap X_s(\mathcal{F}_q)$ for $q\in V_w^\delta$ are diffeomorphic. But F_q^w can be naturally identified with the torus $\mathbb{T}/w$, which follows from writing the equation for $H_s^w(0)$ in the local coordinates $\{y_i\}$ from Lemma 3.9:

$$s^{\lambda(0)}y_1 + P_1(y_2,\dots,y_d) = 0,$$

where $P_1(y_2,\dots,y_d)$ is a Laurent polynomial independent of y_1. Restricting to the fiber $X_s(q)$ means fixing absolute values of $y_i,\ i=2,\dots,d$. A point on the torus $\mathbb{T}/w$ determines the phases of $y_i,\ i=2,\dots,d$. Once $y_i,\ i=2,\dots,d$, are fixed, there is a unique solution to the equation of $H_s^w(0)$.

Thus, $f_s : H_s^{\mathrm{sm}}\to\Sigma\backslash N(D)$ is a torus fibration. The only thing left to check is that it has the correct monodromy.

Note that all diffeomorphisms $F_q\cong F_q^v$, $q\in U_v^\epsilon$, and $F_q\cong F_q^w$, $q\in V_w^\delta$, are deformation diffeomorphisms. Hence, the transitions maps between $\mathbb{T}_v$ and $\mathbb{T}/w$, for $q\in U_v\cap V_w$, are homotopic to the map $f_{vw}:\mathbb{T}_v\to\mathbb{T}/w$. But monodromy is a homotopy invariant, hence, it has to be equal to the one given by the maps f_{vw}. This completes the proof. □

Remark. The same statement was proven in [Zha00] for regular hypersurfaces in smooth toric varieties using partition of unity arguments. This method can also be applied in our situation since we do not touch the singular part of X_{Δ_ν} at all.

Remark. The fiber isomorphisms $F_p \cong \mathbb{T}_v$ depend on the value of the phase of s. If we go around a loop $s \mapsto se^{2\pi i}$, we won't come back to the original diffeomorphism $H_s^{\mathrm{sm}} \to W^\epsilon$. Rather, it will be a composition with a generalized Dehn twist, namely, the diffeomorphism $W^\epsilon \to W^\epsilon$ which is the fiber wise shift by a section of $W^\epsilon \to \Sigma \backslash N(D)$ (the tori are abelian groups). Such a section was explicitly written down in [Zha00].

4. Outlook

4.1. **Combinatorics.** It would be interesting to know whether the subdivision of $|\Sigma|$ given by the $F \times F^\vee$ can be realized as the boundary of a d-dimensional polytope (as the picture suggests). The dual of the face lattice would be given by the lattice of intervals in the face lattice of Δ (or of $\Delta^\vee$).

In fact, if there was a realization of the combinatorial type of Δ with an identification $\mathbb{R}^d \cong (\mathbb{R}^d)^*$ such that the normal cones of F and $F^\vee$ intersect in their relative interiors, then the Minkowski sum $\Delta + \Delta^\vee$ would do the trick. For $d = 3$, Koebe's Theorem [Thu80] guarantees such a realization.

4.2. **The Hausdorff convergence.** There is a natural family of complex structures J_s on the model torus fibration $h\colon W \to \Sigma \backslash D$. Namely, for a given s we can take $dy_i + \frac{\sqrt{-1}}{\log |s|} d\theta_i$ to be the holomorphic 1-forms on W, where $\{y_i\}$ are the affine coordinates on $\Sigma \backslash D$ and $\{\theta_i\}$ are the corresponding coordinates on the torus fibers. Also, given a Riemannian metric $\sum g_{ij} dy_i \otimes dy_j$ on $\Sigma \backslash D$, one can define the Kähler metric on W by $\omega_{ij} = \sum g_{ij}(dy_i + \frac{\sqrt{-1}}{\log |s|} d\theta_i) \otimes (dy_j - \frac{\sqrt{-1}}{\log |s|} d\theta_j)$. If, in addition, g_{ij} satisfy the real Monge-Ampère equation: $\det g_{ij} \equiv 1$, then the induced metric on W is Ricci-flat.

In the second part of the paper we will show that H_s^{sm} embeds into (W, J_s) "almost" holomorphically. Moreover, we will construct a family of $\mathbb{T}$-invariant Kähler forms ω_s on X_{Δ_ν} in the class $\frac{[\nu]}{\log |s|}$, such that the pairs $(H_s, H_s^{\mathrm{sing}})$ with the induced metrics converge in the Gromov-Hausdorff sense to the pair (Σ, D). Here Σ will carry a compact metric space structure which will restrict to a Riemannian metric on $\Sigma \backslash D$.

4.3. **Non-Archimedean geometry.** The family H_s^{aff} can be thought of as an affine algebraic hypersurface in $(K^*)^d$ defined over a complete algebraically closed field K that contains $\mathbb{C}((s))$. Then, as the name suggests, the image of H_s^{aff} under the valuation map $val : (K^*)^d \to R^d$ will be the non-Archimedean amoeba $\mathcal{A}_\infty^\lambda$ (cf. [Kap00]).

Interestingly, the potential functions for our Kähler forms ω_s will come from smoothing a convex piecewise linear function on $\mathbb{R}^d$. This gives another evidence that there may be a reformulation of mirror symmetry purely in non-Archimedean terms. There is a partial understanding of this approach [Kon00], but the entire program still remains wide open.

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Mathematics Department, Duke University, Durham, NC 27708, USA
E-mail address: `[haase,zharkov]@math.duke.edu`