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INTEGRAL AFFINE STRUCTURES ON SPHERES AND TORUS FIBRATIONS OF CALABI-YAU TORIC HYPERSURFACES II

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Abstract. This paper is a continuation of our paper [HZ02] where we have built a combinatorial model for the torus fibrations of Calabi-Yau toric hypersurfaces. This part addresses the connection between the model torus fibration and the complex and Kähler geometry of the hypersurfaces.

1. Introduction

In this paper we endow the topological model Σ constructed in [HZ02] with the structure of a metric space which is a Kähler affine manifold away from the (codimension 2) discriminant locus and relate it to the *geometry* of the Calabi-Yau family of toric hypersurfaces Z_s near large complex structure limit point.

The main result is the following. Given an integral Kähler affine structure on $\Sigma \backslash D$ which is in the right class and satisfies certain (bi-polyhedral) compatibility conditions, we construct a family of Kähler metrics on Z_s such that as $s \to \infty$ (the large complex structure limit):

- The embedding $\psi : Z_s^{\mathrm{sm}} \hookrightarrow W_s$ of "smooth" portions of the hypersurfaces into the model torus bundle (with the right twist) identifies the scalar products and the complex structures on the tangent spaces $T_x Z_s$ and $T_{\phi(x)} W_s$ up to terms of order $o(1)$ uniformly in $x \in Z_s^{\mathrm{sm}}$.
- the Gromov-Hausdorff distance between the pairs $(Z_s, Z_s \backslash Z_s^{\mathrm{sm}})$ and (Σ, D) is of order $o(1)$.

Interchanging the input data (Δ, S, λ) with $(\Delta^\vee, T, \nu)$ and repeating the construction of the Kähler metrics for the *dual* Calabi-Yau family gives rise to the same limiting metric space Σ and the *dual* Kähler affine structure. This result constitutes a significant part of the the limiting mirror symmetry conjecture (cf. [KS01]).

The major missing part toward proving the metric collapse is that our metrics are not Ricci-flat. To establish the Ricci-flatness away from the discriminant one would need the affine Calabi conjecture (see Section 2.3). But even assuming a

Monge-Ampère solution on $\Sigma \backslash D$ the behavior of true Calabi-Yau metrics is not expected to be approximated by a semi-flat construction near the discriminant locus. A further development in this direction requires some local estimates on CY metrics near singularities. We hope to address these issues by using generalized Gibbons-Hawking ansatz elsewhere (cf. [Zha02]).

Notations. We continue to use notations from [HZ02] with some minor modifications.

$(\Delta, S), (\Delta^\vee, T)$ is a dual pair of d-dimensional reflexive polytopes with coherent triangulations of their boundaries.

$\Delta_{\mathbb{Z}}$ is the set of integral points of Δ.

$\mathrm{SC}(S)$ is the secondary cone in $\mathbb{R}^{|\Delta_{\mathbb{Z}}|}$ corresponding to the triangulation $S * \{0\}$ of Δ.

λ and ν are integral vectors in the interiors of the respective secondary cone.

X_T the toric variety associated to the simplicial fan in $(\mathbb{R}^d)^*$ given by the triangulation T.

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2. The model: Kähler affine structures and torus bundles

2.1. **Integral Kähler affine structures.** Let $\mathbb{A}^n$ be n-dimensional affine space. An *integral affine structure* on an n-dimensional manifold Y is given by an open covering $\{U_\alpha\}$ of Y together with coordinates $\phi_\alpha : U_\alpha \to \mathbb{A}^n$ such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are in $\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ on the non-empty overlaps $U_\alpha \cap U_\beta$. An integral *Kähler* affine structure on Y is a Riemannian metric g which is potential in local affine coordinates, i.e. $g_{ij} = \frac{\partial^2 K_\alpha}{\partial y_i \partial y_j}$ for some local potentials K_α.

The dual Kähler affine structure on the same Riemannian manifold (Y, g) is defined as follows (cf [KS01]). We use the same covering $\{U_\alpha\}$. The new affine coordinates are $x_i = \frac{\partial K_\alpha}{\partial y_i}$ which take values in the dual affine space $\mathbb{A}^{n*}$ (the underlying vector spaces for $\mathbb{A}^n$ and $\mathbb{A}^{n*}$ are naturally dual). The new local potentials $\hat{K}_\alpha$ are defined by the Legendre transforms of the old ones:

$$\hat{K}_\alpha = \max_{y \in U_\alpha} \{\langle x, y \rangle - K_\alpha(y)\}.$$

Here one needs to choose origins in $\mathbb{A}^n$ and $\mathbb{A}^{n*}$ to define the pairing. Different choices give rise to equivalent Kähler affine structures.

Given an affine structure on Y one can consider its monodromy representation $\pi_1(Y) \to SL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$. Two equivalent affine structures have conjugate monodromies.

For a Kähler affine manifold (Y, g) one can define (cf. [KS01]) a characteristic class $[g]$ of the metric, which is an analog of the Kähler class in complex geometry. Let $\mathcal{A}f_Y$ be the sheaf of locally affine functions. The metric is given by local potentials in affine coordinates: $g_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}$. Then the differences of the potentials on the overlaps will define a Čech cohomology class $[g] \in H^1(Y, \mathcal{A}f_Y)$.

It is more natural to combine the monodromy representation and the metric class into one class, which we will call the class of *affine polarization*. It can be represented by a Čech 1-cocycle with values in the semi-direct product $(\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n) \ltimes \mathrm{Aff}_n$, where the affine transformations $\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ act on the affine functions Aff_n from the right.

The natural projection onto the normal component $\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ in the above semi-direct product gives the monodromy representation. To recover the metric class, however, one needs to fix a splitting $(\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n) \ltimes \mathrm{Aff}_n \to \mathrm{Aff}_n$. Different splittings will give conjugate metric classes.

For the purposes of this paper we consider a convenient $(n + 2)$-dimensional faithful representation of the group $(\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n) \ltimes \mathrm{Aff}_n$. Let us choose q – an integral vector in $\mathbb{R}^{n+2}$, and p – an integral vector in the dual space $(\mathbb{R}^{n+2})^*$. Then this representation provides an isomorphism of $(\mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n) \ltimes \mathrm{Aff}_n$ with the following subgroup of $\mathrm{GL}_{n+2}(\mathbb{R})$:

$$G_n(p, q) := \{g \in \mathrm{GL}_{n+2}(\mathbb{R}) \ : \ g(q) = q, \ g^*(p) = p, \ g|_{\{\langle p,x\rangle=0\}/q} \text{ is integral}\},$$

where $g^* : (\mathbb{R}^{n+2})^* \to (\mathbb{R}^{n+2})^*$ is the adjoint linear transformation. The affine space $\mathbb{A}^n$ can be identified with $\{\langle p, x\rangle = 1\}/q$. The $G_n(p, q)$ action on it gives the corresponding affine transformation of $\mathbb{A}^n$. To recover the affine function $f : \mathbb{A}^n \to \mathbb{R}$ one needs to fix an integral linear functional $l \in (\mathbb{Z}^{n+2})^*$, such that $l(q) = 1$. Then $f(x) = l(g(x)) - l(x)$ is a function, well defined on the quotient $\{\langle v, x\rangle = 1\}/w$. Only the representing Čech cocycle depends on the choice of l, not the metric class itself.

For the (mirror) symmetry sake we also choose an integral element $k \in \mathbb{R}^{n+2}$ with $p(k) = 1$. The vector k defines an origin in $\mathbb{A}^n$, hence it allows to recover the translational part of the affine transformation. Again, the class of this translational part, called the *radiance obstruction* (cf. [GH84]), is independent of the choice of k (different k' give rise to conjugate monodromies). As was noted in [GS02] the radiance obstruction class is dual to the linear part of the metric class under the duality between the Kähler affine structures.

More generally, it is also clear that the full polarization class for the dual Kähler affine structure can be represented by the adjoint inverse transformations for each $U_\alpha \cap U_\beta$, with the rôles of q, k and p, l exchanged.

Given a basis $\{e_i\}$ of $\mathbb{R}^{n+2}$ such that $\langle p, e_i\rangle = 0, \ i = 1, \dots, n + 1, \ e_{n+1} = q$ and $e_{n+2} = k$, the group $G_n(p.q)$ can be represented by non-degenerate matrices in the

form
$$\begin{pmatrix} A & 0 & b \\ a & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$
where A and b represent the linear and translational parts of the affine transformations, and a, c are the linear and constant parts of the affine function, respectively.

All of the above (including the affine structure itself) can be defined even if we do not require the affine charts to be maps into the *same* affine space. We won't have groups anymore, but in all cocycle conditions the compositions still make sense. In particular, to specify an integral affine structure we would need continuous maps $\phi_\alpha : U_\alpha \to \mathbb{A}^n_\alpha \cong \{\langle p_\alpha, x_\alpha \rangle = 1\}/q_\alpha$ together with integral elements $q_\alpha, k_\alpha \in \mathbb{R}^{n+2}_\alpha$ and $p_\alpha, l_\alpha \in \mathbb{R}^{n+2^*}$ with $p_\alpha(k_\alpha) = 1,\ l_\alpha(q_\alpha) = 1$, such that the transition maps $\phi_{\alpha\beta} : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ satisfy the corresponding invariance, coinvariance and integrality conditions.

The cohomological information, such as monodromy, radiance obstruction and the metric class, is encoded in the transition maps $\phi_{\alpha\beta}$.

2.2. **Bi-polyhedral Kähler affine structures.** We will be interested in a very special types of integral Kähler affine structures. These structures arise in the metric limits of Calabi-Yau hypersurfaces and complete intersections in toric varieties.

Definition. An integral affine structure on Y is *polyhedral* if there is an n-dimensional polyhedral complex P, a collection of disjoint open sets $\{U_\alpha\}$, whose closures cover Y: $Y = \bigcup \overline{U}_\alpha$, and a continuous map $\phi : Y \to P$, which provides an affine homeomorphism of each U_α with the interior of some n-dimensional face of P. We say that the pair $(\{U_\alpha\}, P)$ realizes the polyhedral affine structure if P is minimal, which, in particular, means that there is a bijection between open sets $\{U_\alpha\}$ and n-dimensional cells of P.

Definition. An integral Kähler affine structure on Y is *bi-polyhedral* (bi-PIKAS for short) if there is a bi-partite covering $\{U_\alpha, V_\beta\}$ of Y and two polyhedral complexes $P, \hat{P}$ such that $(\{U_\alpha\}, P)$ and $(\{V_\beta\}, \hat{P})$ provide polyhedral realizations of the underlying affine structure and its dual, respectively. We say that the bi-polyhedral Kähler affine structure is of type $(\{U_\alpha, V_\beta\}, P, \hat{P})$.

The bi-polyhedral property imposes very severe restrictions on the compatibility between Riemannian metric and affine structure. In particular, Hess $K_\alpha \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ and the Cauchy-Schwartz inequality implies that the metric completion of Y can be identified with P or $\hat{P}$. This endows both polyhedral complexes with (the same) structure of complete metric space.

Next we want to show the existence of bipolyhedral integral Kähler affine structures on $\Sigma \backslash D$. Recall from [HZ02] that $\Sigma \backslash D$ has a bipartite covering by open sets U_v and V_w. Also, given vectors λ, ν in the interiors of the respective secondary cones $\mathrm{SC}(S), \mathrm{SC}(T)$ with $\lambda(0) = \nu(0) = 0$ we can define the polytopes

$$\Delta_\lambda^\vee = \{n \in \mathbb{R}^d \ : \ \langle m, n\rangle + \lambda(m) \le 0 \text{ for all } m \in \Delta_{\mathbb{Z}}\},$$
$$\Delta_\nu = \{m \in (\mathbb{R}^d)^* \ : \ \langle m, n\rangle + \nu(n) \le 0 \text{ for all } n \in \Delta_{\mathbb{Z}}^\vee\}.$$

In order to specify a bipolyhedral Kähler affine structure on $\Sigma\backslash D$ associated with the vectors λ, ν we will provide the following data. A Legendre dual pair of convex functions $\Phi, \hat{\Phi}$ on $\Delta_\lambda^\vee, \Delta_\nu$, respectively, smooth on each strata of the respective polytope. (This implies that the Hessians of both $\Phi, \hat{\Phi}$ are positive along the strata.) Then the restrictions of Φ to the facets of $\Delta_\lambda^\vee$ serve as potentials for the metric along U_v. For future use we will prove that it is possible to choose these functions consistently in λ and ν.

Definition. Suppose, for each pair $(\lambda, \nu) \in \mathrm{SC}(S) \times \mathrm{SC}(T)$ we have a bipolyhedral Kähler affine structure on $\Sigma\backslash D$ of type (λ, ν), which varies continuously with (λ, ν) in the Hausdorff topology of metric structures on Σ. We call such a family *projective* if:

- For any linear functions $l \in \mathbb{R}^d$, $l^\vee \in (\mathbb{R}^d)^*$, the bi-PIKAS for $(\lambda + l, \nu + l^\vee) \in \mathrm{SC}(S) \times \mathrm{SC}(T)$ have the same underlying Kähler affine structure.
- The bi-PIKAS for $(\epsilon^{-1}\lambda, \epsilon\nu)$ differs from the bi-PIKAS for (λ, ν) by the ϵ-rescaling

$$(\lambda, \nu, K_\alpha(y_\alpha), \hat{K}_\alpha(x_\alpha), g_{ij}) \mapsto (\epsilon^{-1}\lambda, \epsilon\nu, K_\alpha(\epsilon y_\alpha), \hat{K}_\alpha(\epsilon^{-1}x_\alpha), \epsilon^2 g_{ij}) \tag{1}$$

Note here that adding global linear functions to the potentials $\Phi, \hat{\Phi}$ will induce translations of the polytopes $\Delta_\nu, \Delta_\lambda^\vee$. Though giving different bi-PIKAS (as we defined them) this will have no effect on the underlying Kähler affine structures (the latter will be canonically equivalent).

Another important observation is that rescaling the data for bi-PIKAS according to (1) will provide the same metric on $\Sigma\backslash D$, though different affine structures.

Proposition 2.1. *There are Legendre dual functions $\Phi \colon \Delta_\lambda^\vee \to \mathbb{R}$ and $\hat{\Phi} \colon \Delta_\nu \to \mathbb{R}$ that define a projective family of bi-PIKAS on $\Sigma\backslash D$.*

Proof. First, we choose a smooth function with positive Hessian on $\Delta_{\lambda-\beta}^\vee$ whose gradients stay in $\Delta_{\nu-\beta}$, and cover $\Delta_{\nu-2\beta}$.

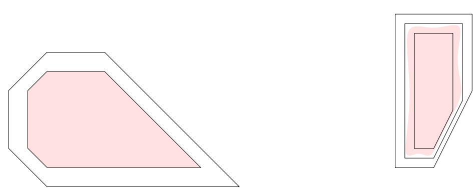

FIGURE 1: The domain for the first step. The set of gradients.

As a second step, we need a continuous strictly convex function $\tilde{\nu}$ on $\beta\Delta^\vee$ that is an approximation of the function ν with the following properties.

- $\tilde{\nu}$ is piecewise smooth.
- $\operatorname{Hess}\tilde{\nu} > 0$ on the smooth pieces.
- The gradients along $\operatorname{cone}\tau$ belong to a neighborhood of the corresponding vertex in Δ_ν that are pairwise disjoint, and do not meet the β neighborhood of the barycenter of Δ_ν.
- For a vertex $w \in T$, the w-directional derivatives equal $-\nu(w)$ in the star neighborhood of βw in the barycentric subdivision of βT.
- Furthermore, the second derivative along the ray through w vanishes at w. THIS HAS THE EFFECT THAT THERE ARE MORE PIECES WHERE THIS HAPPENS, WHICH IN TURN IMPLIES THAT THE LEGENDRE DUAL IS NOT C^1

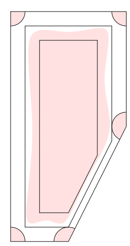

FIGURE 2: The set of gradients of $\tilde{\nu}$.

We obtain a convex function on $(\mathbb{R}^d)^*$ if we consider the lower hull of the $(d+1)$-dimensional Minkowski sum of graphs of the two functions.

Finally, we want to obtain a function that is smooth along the strata. As explained in § 3.2, we convolute with a kernel that depends on the position $x \in \Delta_\lambda^\vee$ as follows. Consider the quadratic form

$$Q(x) = \sum_{v \in S} Q_v(x) \quad \text{where} \quad Q_v(x) = \frac{1}{\rho_v(x)} v^2$$

This quadratic form is non-degenerate because the v's span $(\mathbb{R}^d)^*$. It has a dominant summand if x is close to a facet. Now our kernel will be a normalized $e^{1-Q(x)}$. Its support – the ellipsoid given by $Q(x) \leq 1$ – depends on the position as sketched in the figure.

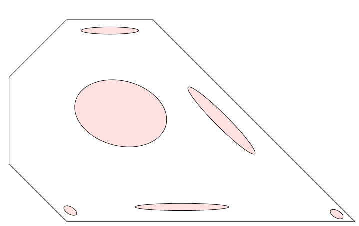

FIGURE 3: The support of the mollifier.

The obtained function will have a positive Hessian along the strata, so that the Legendre dual function will be smooth as well. □

In the future we will abbreviate the type of a bipolyhedral integral Kähler affine structure on $\Sigma\backslash D$ by simply (λ, ν) having fixed the covering $(\{U_v, V_w\})$.

Applying Proposition 2.1 allows us to define projective families of bi-PIKAS parameterized by pairs $(\lambda, \nu) \in \mathrm{SC}(S) \times \mathrm{SC}(T)$.

The metric completion of $\Sigma\backslash D$ can be identified with Σ and endowed with the structure of a compact metric space via the bipolyhedral homeomorphisms $\phi, \hat{\phi}$:

(2)
$$\begin{tikzcd} & \Sigma \arrow[dl, "\phi"'] \arrow[dr, "\hat{\phi}"] & \\ \partial\Delta_\lambda^\vee & & \partial\Delta_\nu \end{tikzcd}$$

Throughout the paper we will often identify points in Σ, $\partial\Delta_\lambda^\vee$ and $\partial\Delta_\nu$ by means of these homeomorphisms when there is no confusion.

We can realize the affine coordinates y on U_v and V_w explicitly as taking values in the following (affine) subspaces and quotients of $\mathbb{R}^d$:

$$y(q) \in \mathbb{R}^d_v(\lambda(v)) := \{n \in \mathbb{R}^d \ : \ \langle v, n\rangle + \lambda(v) = 0\}, \quad q \in U_v,$$
$$y(q) \in \mathbb{R}^d/w, \quad q \in V_w,$$

and the transition maps are given by the obvious projections $\mathbb{R}^d_v(\lambda(v)) \to \mathbb{R}^d/w$. Then the affine monodromy (cf. [HZ02, Lemma 2.4]) along a primary loop $(v_0w_0v_1w_1)$ is given by:

(3)
$$n \mapsto n + [\langle v_1, n\rangle + \lambda(v_1)](w_1 - w_0), \quad n \in \mathbb{R}^d_v(\lambda(v)).$$

To describe the full polarization class $[\lambda, \nu]$ of a bipolyhedral Kähler affine structure on $\Sigma\backslash D$ of type (λ, ν) we consider the representation of $(\mathrm{SL}(d-1, \mathbb{Z}) \ltimes \mathbb{R}^{d-1}) \ltimes \mathrm{Aff}_{d-1}$ in $\mathbb{R}^d \oplus \mathbb{R}$. The dual space is identified with $(\mathbb{R}^d)^* \oplus \mathbb{R}$. For the charts U_v and V_w we set

$$U_v: \ p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ k = \begin{pmatrix} k_v \\ 0 \end{pmatrix}, \ q = (v, 0), \ l = (0, 1),$$
$$V_w: \ p = \begin{pmatrix} w \\ 0 \end{pmatrix}, \ k = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ q = (0, 1), \ l = (l_w, 0),$$

where we have chosen integral elements $l_w \in (\mathbb{Z}^d)^*$ and $k_v \in \mathbb{Z}^d$ such that $\langle l_w, w\rangle = 1$ and $\langle v, k_v\rangle = 1$. Then for $U_v \cap V_w$ the cocycle transformation $g_{vw} : \mathbb{R}^d \oplus \mathbb{R} \to \mathbb{R}^d \oplus \mathbb{R}$ is given by:

$$\begin{pmatrix} n \\ s \end{pmatrix} \mapsto \begin{pmatrix} n' - [1 + \nu(w)] \langle l_w, n'\rangle w + sw \\ \langle v, n\rangle \end{pmatrix},$$

where $n' = n - \langle v, n \rangle [1 + \lambda(v)] k_v$. The cocycle $\{g_{vw}\}$ represents the polarization class which we will denote by $[\lambda, \nu]$.

Then the monodromy representation $\pi_1(\Sigma \backslash D) \to \mathrm{GL}(\mathbb{R}^d \oplus \mathbb{R})$ along a primary loop $(v_0 w_0 v_1 w_1)$ is given by

$$\begin{pmatrix} n \\ s \end{pmatrix} \mapsto \begin{pmatrix} n \\ s \end{pmatrix} + \alpha(n) \begin{pmatrix} w_1 - w_0 \\ \nu(w_1) - \nu(w_0) \end{pmatrix},$$

where $\alpha(n) = v_1(n) + \lambda(v_1) v_0(n) - [1 + \lambda(v)] v_1(k_{v_0}) v_0(n)$. Considered on the quotient by the last coordinate and using k_{v_0} to identify $\mathbb{A}^{d-1}$ and $\mathbb{R}^d_v(\lambda(v))$ via

$$\begin{gathered} \{\langle v, n \rangle = 1\} \to \{\langle v, n \rangle + \lambda(v) = 0\} \\ n \mapsto n - \langle v, n \rangle [1 + \lambda(v)] k_v, \end{gathered}$$

this recovers the affine monodromy (3) on $\mathbb{R}^d_v(\lambda(v))$.

Following through the above calculation shows that the converse is also true: any bipolyhedral Kähler affine structure on $\Sigma \backslash D$ in the class $[\lambda, \nu]$ is, in fact, of type $(\{U_v, V_w\}, \partial\Delta^\vee_\lambda, \partial\Delta_\nu)$.

2.3. **The Calabi conjecture.** Among all bi-polyhedral Kähler affine structures of type (λ, ν) we expect to find a unique distinguished representative – the Monge-Ampère structure: in affine coordinates the metric satisfies $\det g_{ij} = c$. Its metric completion to Σ is supposed to be the limit of the Ricci-flat metrics on the families of Calabi-Yau hypersurfaces.

Conjecture 2.2 (cf. also [KT02])**.** *There is a unique bi-polyhedral Kähler affine structure on $\Sigma \backslash D$ of type (λ, ν) such that the metric is Monge-Ampère:* $\det g_{ij} = (\mathrm{Vol}\, \partial\Delta^\vee_\lambda)^{-1} \cdot \mathrm{Vol}\, \partial\Delta_\nu$.

Note that the Monge-Ampère constant c is determined from calculating the metric volume of Σ as $\sqrt{c} \cdot \mathrm{Vol}\, \partial\Delta^\vee_\lambda = \sqrt{c^{-1}} \cdot \mathrm{Vol}\, \partial\Delta_\nu$, where Vol means the affine volume of the corresponding polytopal complex. Also note that the rescaled data $(\epsilon^{-1}\lambda, \epsilon\nu, K_\alpha(\epsilon y_\alpha), \hat{K}_\alpha(\epsilon^{-1} x_\alpha), \epsilon^2 g_{ij})$ provides the Monge-Ampère structure in the class $[\epsilon^{-1}\lambda, \epsilon\nu]$ with the same metric on Σ. Thus the Monge-Ampère bi-PIKAS fit together into a projective family.

We will not address this conjecture any further here, rather we will be happy to start with any bi-polyhedral Kähler affine structure on $\Sigma \backslash D$ given by a pair $(\Phi, \hat{\Phi})$.

2.4. **The model torus fibrations as Kähler manifolds.** The $(d-1)$-torus fibration $W(\lambda, \theta)$ (more naturally, a $\mathbb{T}^{d-1}$-torsor) will depend on additional phase multi-parameter $\theta := \{\theta_v\}, v \in \mathrm{vert}(S)$, where all θ_v have values in $\mathbb{R}/\mathbb{Z}$. First, we form (trivial) affine torus bundles over the affine open sets by identifying the fibers with the affine tori

$$\mathbb{T}_v(\theta_v) := \{n \in \mathbb{T} \ : \ \langle v, n\rangle + \theta_v \equiv 0 \mod \mathbb{Z}\} \quad \text{over } U_v,$$
$$\mathbb{T}/w := (\mathbb{R}^d/w)/(\mathbb{Z}^d/w) \quad \text{over } V_w.$$

The gluing maps are independent of a base point in an overlap $U_v \cap V_w$ and given there by the natural projection $\mathbb{T}_v(\theta_v) \to \mathbb{T}/w$. This defines the torsor $\pi : W(\lambda,\theta) \to \Sigma\backslash D$.

The topology of the total space of the torsor is determined by the combinatorics of Σ, i.e., independent of λ and θ as long as λ is in the right secondary cone $\mathrm{SC}(S)$. In particular, all $W(\lambda,\theta)$ are diffeomorphic to each other, though not canonically.

Since the linear parts of the transition maps are the same for the base and for the fibers, the tangent space at any point in $W(\lambda,\theta)$ splits canonically as

$$lT_{W(\lambda,\theta)} \cong T_{\Sigma\backslash D} \oplus T_{\Sigma\backslash D}.$$

This allows to define a canonical (integrable) almost complex structure on $W(\lambda,\theta)$ as

$$J_w = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

Given a Riemannian metric g_{ij} on Y, one can define the pullback metric $\pi^*(g_{ij})$ on $W(\lambda,\theta)$ which is, in fact, Kähler. If, in addition, g_{ij} satisfy the real Monge-Ampère equation, the induced metric on $W(\lambda,\theta)$ is Ricci-flat.

There is another slightly different description of the torus bundle over a Kähler affine manifold Y (cf. [KS01]), which is useful when considering limiting behavior of Calabi-Yau degenerations.

We define a torus fibration $W_\epsilon(\lambda)$ as the quotient of the total space of the tangent bundle $T(\Sigma\backslash D)$ by the integral lattice spanned by $\left\{\epsilon\frac{\partial}{\partial y_i}\right\}$, where y_i are the affine coordinates. This torus bundle carries canonical complex structure and the pullback metric which comes from the splitting $T_{W_\epsilon(\lambda)} \cong T_{\Sigma\backslash D}\oplus T_{\Sigma\backslash D}$ as before.

But in order to make a connection with the previous picture and with the geometry of toric hypersurfaces we need to twist the complex structure on $W_\epsilon(\lambda)$ by the element of $H^1(\Sigma\backslash D, \mathbb{T}^{d-1})$ associated with the phase parameters θ_v. The resulting Kähler manifold $W_\epsilon(\lambda,\theta)$ can be canonically identified with $W(\epsilon^{-1}\lambda,\theta)$ constructed by the first method starting with the ϵ-rescaled Kähler affine structure (1).

3. Vector fields, foliations and Kähler potentials

This section describes two important ingredients for a later consideration of the geometry of toric hypersurfaces. The first is the foliations of the amoebas which will induce the torus fibrations of the hypersurfaces. The second is a Kähler potential on the toric variety which induces the metric.

From now on we will fix a projective family of bi-PIKAS on $\Sigma\backslash D$. We will refer to a member of type (λ, ν) as the (λ, ν)-bi-PIKAS. Also we will fix some linear functional ℓ positive on the secondary cone $\mathrm{SC}(S)$. This will allow us to talk about the scale $\ell(\lambda)$ of the vector λ. We will abuse the notation and denote the analogous scale for ν by $\ell(\nu)$.

Let denote by $\overrightarrow{m}$ the vector in $\mathbb{R}^{\partial\Delta_{\mathbb{Z}}}$ with $\overrightarrow{m}(m) = 1$ and 0 otherwise, and let $\overrightarrow{\mathbb{1}} := \sum_{m\in\partial\Delta_{\mathbb{Z}}} \overrightarrow{m}$. For a real number β we will often write $\lambda+\beta$ meaning $\lambda+\beta\cdot\overrightarrow{\mathbb{1}}$. We will say that $\beta > 0$ is *small in the λ-scale*, or simply λ-small, if the vectors $\lambda + \beta(\sum \pm\overrightarrow{m_i})$, for all possible collections $\{m_i\} \subset \partial\Delta_{\mathbb{Z}}$, are still in the interior of the secondary cone $\mathrm{SC}(S)$. And similar for the ν-scale.

3.1. **Neighborhoods of the discriminant.** For a given (λ, ν) we define subsets $U_v^\beta \subset U_v$ and $V_w^{\beta^\vee} \subset V_w$ whose union will give the complement of a neighborhood of D depending on two real parameters $\beta, \beta^\vee > 0$. We assume $\beta, \beta^\vee$ to be small in the λ, ν scales, respectively. In what follows we identify Σ with $\partial\Delta_\lambda^\vee$ and $\partial\Delta_\nu$ via the maps ϕ and $\hat{\phi}$ associated with the bi-PIKAS of type (λ, ν).

For $v \in \mathrm{vert}(S)$ we let U_v^β be the set of points in the corresponding facet of $\Delta_\lambda^\vee$ which lie in the closed polyhedron $Q^\lambda_{(v|\{0\})}(\beta)$ (cf. [HZ02, Section 3.2]), and similar for $V_w^{\beta^\vee}$. Explicitly,

$$U_v^\beta := \{n \in U_v \ : \ \langle m, n\rangle + \lambda(m) \le -\beta, \text{ all } m \in \Delta_{\mathbb{Z}}\backslash\{v, 0\}\},$$
$$V_w^{\beta^\vee} := \{m \in V_w \ : \ \langle m, n\rangle + \nu(n) \le -\beta^\vee, \text{ all } n \in \Delta_{\mathbb{Z}}^\vee\backslash\{w, 0\}\}.$$

Because $\beta, \beta^\vee$ are small the sets U_v^β and $V_w^{\beta^\vee}$ are non-empty. We define the smooth part of Σ as

$$\Sigma^{\mathrm{sm}} := \bigcup_{v\in\mathrm{vert}(S)} U_v^\beta \cup \bigcup_{w\in\mathrm{vert}(T)} V_w^{\beta^\vee}.$$

Then the neighborhood of D is defined as the complement to all these closed sets in Σ:

$$N_{\lambda,\nu}^{\beta,\beta^\vee}(D) := \Sigma\backslash\Sigma^{\mathrm{sm}}.$$

An important observation is that $N_{\lambda,\nu}^{\beta,\beta^\vee}(D) \to D$ as $\beta, \beta^\vee \to 0$ in the scales of λ, ν, respectively.

3.2. **Regularization.** We will be smoothing various functions later on in this section so let us recall the standard regularization techniques. Given a convex bounded domain $P \subset \mathbb{R}^n$ with piece-wise smooth boundary let ρ be a mollifier whose support is P, i.e. a positive $\mathbb{C}^\infty(\mathbb{R}^n)$ function vanishing exactly outside P and such that $\int_{\mathbb{R}^n} \rho dx = 1$. For instance, if P is a polytope in $\mathbb{R}^n$ given by a collection of inequalities $\{\langle v_i, x\rangle + \lambda_i \le 0\}$ one can take the usual bell-shaped function

$$\rho = c\prod_i \rho_i, \qquad \text{where } \rho_i(x) = \begin{cases} e^{\frac{1}{\langle v_i, x\rangle + \lambda_i}}, & \langle v_i, x\rangle + \lambda_i \leq 0 \\ 0, & \langle v_i, x\rangle + \lambda_i \geq 0 \end{cases},$$

and the constant c is determined from the normalization. Set $\rho_h := \frac{1}{h^n}\rho(\frac{x}{h})$. For any locally integrable function $u \in L^1_{loc}(\mathbb{R}^n)$ we can apply the standard regularization procedure by taking the convolution with ρ_h:

$$u_h(x) := \rho_h * u = \int_{\mathbb{R}^n} \rho_h(x-y)u(y)\,dy.$$

Then, if $u \in C^p(\mathbb{R}^n)$, the functions u_h are $\mathbb{C}^\infty$ and approaching u as $h \to 0$ in the C^p-norm uniformly on any compact in $\mathbb{R}^n$.

Below we list some elementary properties of u_h which will be useful later.

Proposition 3.1. *Let Ω be a convex domain in $\mathbb{R}^n$, then*

- *u_h is linear in a v-direction in Ω if u is linear in the v-direction in the Minkowski sum $\Omega + h(-P)$, with $\langle v, \nabla u_h\rangle = \langle v, \nabla u\rangle$.*
- *u_h is convex in $\mathbb{R}^n$ if u is. The gradient $\nabla u_h(x), x \in \Omega$, is always inside the convex hull of all possible gradients of u in $\Omega + h(-P)$.*
- *u_h is strictly convex in $\Omega + h(-P)$ if u is convex in $\mathbb{R}^n$ and strictly convex in Ω.*

Proof. All statements are simple consequences of the following observation. The convolution $\rho_h * u$ is a weighted averaging of u over the (translated) support of ρ. The same holds for all derivatives of u as well. □

One can also apply the regularization procedure by taking the convolution with the mollifier parameter depending (smoothly) on the point $x \in \mathbb{R}^d$. That is $u_h(x) := \int_{\mathbb{R}^n} \rho_{h(x)}(x-y)u(y)\,dy$. Or, even, more generally the entire shape of the support of the mollifier can smoothly depend on the center of convolution. We have used this technique of varying support to prove the existence of bi-PIKAS in Proposition 2.1.

3.3. **Fibration.** We will construct a foliation of $\mathbb{R}^d \backslash \Delta^\vee_\lambda$ by straight lines/rays by specifying a "convex" vector field on $\partial\Delta^\vee_\lambda$, which is smooth in Σ^{sm}.

Recall that the piece-wise linear functions $L_\nu : (\mathbb{R}^d)^* \to \mathbb{R}$ and $L_\lambda : \mathbb{R}^d \to \mathbb{R}$, the Legendre transforms of ν and λ, were defined as

$$L_\nu(m) := \max_{n\in\Delta^\vee_{\mathbb{Z}}}\{\langle m,n\rangle + \nu(n)\}, \quad L_\lambda(n) := \max_{m\in\Delta_{\mathbb{Z}}}\{\langle m,n\rangle + \lambda(m)\}.$$

We fix a mollifier $\rho^\vee$ on $(\mathbb{R}^d)^*$ with support in Δ and consider smooth functions $L_{\nu,h^\vee} := \rho^\vee_{h^\vee} * L_{\nu - h^\vee}$. From the properties of regularization (for $h^\vee$ small in the ν-scale) the slopes $\nabla L_{\nu,h^\vee}(x)$ always lie in $\partial\Delta^\vee$, for any x not in the interior of Δ_ν. In particular, the gradient of $L_{\nu,h^\vee}$ gives a map $\nabla L_{\nu,h^\vee} : \partial\Delta_\nu \to \partial\Delta^\vee$.

Now we can use the identification of $\partial\Delta_\nu$ with $\partial\Delta_\lambda^\vee$ via $\phi\hat{\phi}^{-1}$ given by the (λ,ν)-bi-PIKAS to define a $\partial\Delta^\vee$-valued vector field $\mathfrak{X}_{h^\vee}$ on $\partial\Delta_\lambda^\vee$ by

$$\mathfrak{X}_{h^\vee}(q) := \nabla L_{\nu,h^\vee}(\phi\hat{\phi}^{-1}q).$$

We can extend this vector field $\mathfrak{X}_{h^\vee}$ to $\mathbb{R}^d\backslash\Delta_\lambda^\vee$ using the following identification of $\partial\Delta_\nu$ with $\partial\Delta_{\lambda+t}^\vee$, for any $t\geq 0$. If $\hat{\Phi}$ is the (dual) (λ,ν)-bi-PIKAS potential, then $\hat{\Phi}+tL_{\nu,h^\vee}$ is a strictly convex function on Δ_ν. In particular, its gradient defines a bijection $\nabla\hat{\Phi}+t\nabla L_{\nu,h^\vee}:\partial\Delta_\nu\to\partial\Delta_{\lambda+t}^\vee$. We can write the vector field $\mathfrak{X}_{h^\vee}$ on $\mathbb{R}^d\backslash\Delta_\lambda^\vee$ explicitly, using the fact that

$$L_\lambda(x)=t\Leftrightarrow x\in\partial\Delta_{\lambda+t}^\vee,\ \text{for any } t\geq 0.$$

Namely,

$$\mathfrak{X}_{h^\vee}(x) := \nabla L_{\nu,h^\vee}([\nabla\hat{\Phi}+L_\lambda(x)\nabla L_{\nu,h^\vee}]^{-1}(x)).$$

We summarize properties of this vector field in the following lemma:

Lemma 3.2. *For any $h^\vee>0$, small in the ν-scale, the vector field $\mathfrak{X}_{h^\vee}$ induces a (straight line/ray) foliation $\mathcal{F}_{h^\vee}$ of $\mathbb{R}^d\backslash\Delta_\lambda^\vee$, smooth over $\Sigma\backslash D$, such that*

1. *$\mathfrak{X}_{h^\vee}(q)=w$ for $q\in V_w^{\beta^\vee}$.*
2. *The value of $\mathfrak{X}_{h^\vee}(q)$ is in $(\operatorname{carrier}\sigma)^\vee\subset\partial\Delta^\vee$ for $q\in F_\sigma\subset\partial\Delta_\lambda^\vee$. In particular, $\langle v,\mathfrak{X}_{h^\vee}\rangle=1$ in U_v.*
3. *For any $q\in\Sigma\backslash D$, $|\nabla\mathfrak{X}_{h^\vee}(q)|\leq\frac{C|g_{ij}(q)|}{h^\vee}$, where C is a constant independent of λ and ν, and the gradient and the metric g_{ij} are taken in affine coordinates.*

Proof. From the definition it is easy to see that $\mathfrak{X}_{h^\vee}(q+t\mathfrak{X}_{h^\vee}(q))=\mathfrak{X}_{h^\vee}(q)$, for $q\in\partial\Delta_\lambda^\vee$, which immediately implies the straight ray foliation.

Note that as $h^\vee\to 0$, $L_{\nu,h^\vee}$ converges to L_ν uniformly in $(\mathbb{R}^d)^*$. For $h^\vee=0$ the (discontinuous) vector field $\mathfrak{X}$ has (discrete) values in $\operatorname{vert}(T)$, and (1) and (2) follow immediately from the combinatorics of Σ. They remain true after regularization as well, which is guaranteed by the Proposition 3.1.

The bound (3) on the derivatives of $\mathfrak{X}$ follows from a standard estimate for regularization of piece-wise smooth function L_ν. In the *dual* affine coordinates $\operatorname{Hess} L_\nu$ is a Dirac δ-like distribution supported on $\partial\mathcal{V}$. The norm of its convolution with ρ is bounded by $\frac{C}{(h^\vee)^k}$, where k is the codimension of the support. The constant C takes into account the combinatorics of the polytope Δ, the particular form of the mollifier ρ and the choice of the norm on $\mathbb{R}^{d-1}\cong T_q(\Sigma\backslash D)$. The metric g_{ij} appears from the chain rule: $\nabla\mathfrak{X}_{h^\vee}=g\cdot\operatorname{Hess} L_{\nu,h^\vee}$.

Finally, the smoothness of the vector field $\mathfrak{X}_{h^\vee}$, and hence the smoothness of the foliation $\mathcal{F}$, follow from smoothness of the map $\phi\hat{\phi}^{-1}$ on $\Sigma\backslash D$. $\square$

3.4. **Kähler metrics on the toric variety.** Next we regularize the C^1-potential Φ to get a smooth convex function on $\mathbb{R}^d$ which we will use later on to define a Kähler potential on the toric variety X_T. Let ρ be a mollifier with support in $-\Delta^\vee$. We define $\Phi_h^{\mathrm{sm}} := \rho_h * \Phi$.

Remark. The constructed vector field, foliation, potential, etc., depend on the pair (λ, ν), as well as on the regularization parameters $h, h^\vee$. But to simplify the notations for $\mathfrak{X}, \mathcal{F}, \Phi, \Phi^{\mathrm{sm}}$ we will often leave only those indices which are important in a current consideration and omit the rest when there is no confusion possible.

Before constructing a Kähler potential on the toric variety X_T we need another technical statement.

Lemma 3.3. *For $\tau \in T$, the τ-slope of Φ_h^{sm} is equal to the τ-slope of $-\nu|_\tau$ in some translation $R_{\geq\tau}$ of $\cup_{\tau'\geq\tau}\operatorname{cone}(\tau')$.*

Proof. Consider $h = 0$ first. For a simplex $\tau \in T$, let F_τ be the corresponding face of Δ_ν, and we set $R_\tau := \phi\hat{\phi}^{-1}(F_\tau) + \operatorname{cone}(\tau)$. Then the set $\cup_{\tau'\geq\tau}R_{\tau'}$ contains some translation $R_{\geq\tau}$ of $\cup_{\tau'\geq\tau}\operatorname{cone}(\tau')$.

On the other hand, Φ is the Legendre transform of $\hat{\Phi}|_{\Delta_\nu}$. Hence, if $\nabla\Phi(x) = y$ and r is in the normal cone to Δ_ν at $y \in \Delta_\nu$, then $\nabla\Phi(x+r) = \nabla\Phi(x) = y$. So we see that for $x \in R_\tau$ the gradient $\nabla\Phi(x)$ takes values in the face F_τ of Δ_ν because x. In particular, the τ-slopes of Φ are equal to $-\nu|_\tau$.

For $h > 0$ the statement of the lemma follows from the case $h = 0$ and the Proposition 3.1. The translated cones $R_{\geq\tau}$ become shifted into their interiors by some vectors of size h. □

Now we can use Φ^{sm} with any $h > 0$ to define a Kähler potential on X_T.

Proposition 3.4. *The $(1,1)$-form defined on $(\mathbb{C}\backslash\{0\})^d$ by*

$$\eta := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi^{\mathrm{sm}}\left(\log|x|\right)$$

extends to a smooth (in the orbifold sense) non-negative definite $(1,1)$-form on X_T in the cohomology class $[\eta] = [\nu]$.

Proof. First, we rewrite the form η on $(\mathbb{C}\backslash\{0\})^d$ as

$$\eta = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Phi^{\mathrm{sm}}\left(\log|x|\right) = \frac{1}{2\pi}\left\langle d\left(\nabla\Phi^{\mathrm{sm}}\left(\log|x|\right)\right) \wedge d\operatorname{Arg}(x)\right\rangle,$$

where "$\langle \wedge \rangle$" means also the $\langle\ ,\ \rangle$-pairing between the $(\mathbb{R}^d)^*$-valued gradient $\nabla\Phi^{\mathrm{sm}}$ and the $\mathbb{R}^d$-valued 1-form $d\operatorname{Arg}(x)$.

For a simplex $\tau \in T$ we want to show that η extends to the toric subvariety Z_τ associated to τ. If X_T is smooth, then in a neighborhood of Z_τ we can choose the coordinates similar to those from [HZ02, Lemma 3.9]. That is, we choose a basis $\{e_i\}$ such that

$$\langle e_i, w_j\rangle = -\delta_{ij}, i = 1, \ldots, \dim\tau + 1 \text{ and } \langle e_i, \tau\rangle = 0, i = \dim\tau + 2, \ldots, d.$$

Then, in the coordinates $y_i = x^{e_i}$ the equations for the subvariety $Z_\tau \subset X_T$ are $y_i = 0, i = 1, \ldots, \dim\tau + 1$.

According to the theory of toric varieties (cf., e.g. [Ful93]) a neighborhood of the toric subvariety Z_τ lies in the closure of $\log^{-1}(R_{\geq\tau})$, where $R_{\geq\tau}$ is any translation of the cone $\cup_{\tau'\geq\tau}\operatorname{cone}(\tau')$. But by the Lemma 3.3 the directional derivatives $\langle\nabla\Phi^{\mathrm{sm}}, w_i\rangle$, $w_i \in \tau$, are constant in some translation of $\cup_{\tau'\geq\tau}\operatorname{cone}(\tau')$. Hence, in a neighborhood of Z_τ the form η_a written in the above coordinates is independent of $y_i, i = 1, \ldots, \dim\tau + 1$, and, thus, can be extended to Z_τ.

In case when X_T is an orbifold we may not be able to choose an integral basis $\{e_i\}$ with the above conditions. This corresponds to the fact that we may need to go to a finite cover to get a smooth form by weakening the first set of conditions to be $\langle e_i, \tau\rangle \in \mathbb{Z}$. But the rest of the argument goes through.

Finally, the cohomology class of a $\mathbb{T}$-invariant $(1,1)$-form on a complete toric variety is determined by the image of its moment map. But the moment map for η is given on $(\mathbb{C}\backslash\{0\})^d$ by

$$\mu(x) = \nabla\Phi^{\mathrm{sm}}\left(\log|x|\right),$$

whose extention to the whole toric variety X_T has the image Δ_ν. Hence the class of η is $[\nu]$. □

Finally we can add to η a (small) positive multiple of a canonical (e.g., Fubini-Study) form ω_0. Thus we get a true Kähler form $\omega = \eta + \epsilon\omega_0$ on X_T in the class $[\nu] + \epsilon[\omega_0]$).

4. Geometry of Calabi-Yau toric hypersurfaces

A Calabi-Yau hypersurface Z_a is given by the closure of the set

$$Z_a^{\mathrm{aff}} := \{x \in (\mathbb{C}\backslash\{0\})^d \ : \sum_{m\in\Delta_{\mathbb{Z}}\backslash\{0\}} a_m x^m = 1\}$$

in the toric variety X_T. For an element $z = \{z_1, \ldots, z_n\} \in \mathbb{C}\backslash\{0\}^n$ we will use the notations

$$\log|z| := \{\log|z_1|, \ldots, \log|z_n|\}, \qquad \operatorname{Arg}(z) := \frac{1}{2\pi}\{\arg(z_1), \ldots, \arg(z_n)\}.$$

From now on we set $\lambda := \log|a|$ and require it to be in a proper subcone $\mathrm{SC}^\circ(S)$ of the secondary cone $\mathrm{SC}(S)$. Also, for non-zero a_v, we set $\theta_v := \frac{1}{2\pi}\arg(a_v)$.

4.1. **An embedding of Z_a^{sm} into the model torus bundle.** In [HZ02] we have used the GKZ machinery [GKZ94] for the monomial estimates in the equation of Z_a to establish an embedding of Z_a^{sm} into $W_a := W(\log|a|, \mathrm{Arg}(a))$. The same estimates can be used to find bounds on the discrepancy of this embedding from being holomorphic and isometric. Establishing these bounds will occupy the rest of the section.

To define the fibration we assume that $\lambda = \log|a|$ is sufficiently far in the interior of $\mathrm{SC}(S)$, so that $c := \log|\#\Delta_{\mathbb{Z}}|$ is small in the λ-scale.

Lemma 4.1. *The amoeba $\mathcal{A}^\lambda = \log(Z_a^{\mathrm{aff}})$ lies outside of $\Delta^\vee_{\lambda-c}$. In particular, the foliation $\mathcal{F}_{\lambda-c,\nu,h}$ of $\mathbb{R}^d\backslash\Delta^\vee_{\lambda-c}$ induces the fibration $\mathcal{A}^\lambda \to \partial\Delta^\vee_{\lambda-c}$ by projection along the leaves.*

Proof. Note that for any $m \in \Delta_{\mathbb{Z}}\backslash\{0\}$, if $\langle m, \log|x|\rangle + \log|a| \le -\log|\#\Delta_{\mathbb{Z}}|$, then $|a_m x^m| \le \frac{1}{\#\Delta_{\mathbb{Z}}}$. Hence the equation $\sum_{m\in\Delta_{\mathbb{Z}}\backslash\{0\}} a_m x^m = 1$ cannot have solutions for $x \in \log^{-1}(\Delta^\vee_{\lambda-c})$. □

From now on we will use the $(\lambda - c, \nu)$ bi-PIKAS to identify Σ with $\partial\Delta^\vee_{\lambda-c}$ and fix the vector field and the foliation in $\mathbb{R}^d\backslash\Delta^\vee_{\lambda-c}$. With this identification, given a subset $U \in \Sigma$ we denote by $X(U)$ the closure of the set $\log^{-1}(\cup_{q\in U}\mathcal{F}_q)$ in the toric variety X_T (cf. [HZ02]). Then the smooth part Z_a^{sm} of the hypersurface is defined as:

$$Z_a^{\mathrm{sm}} := Z_a \cap X(\Sigma\backslash N^{\beta,\beta^\vee}_{\lambda-c,\nu}(D)).$$

We define the map ψ over the charts $V_w^{\beta^\vee}$ as the restriction to Z_a of the quotient map:

$$x \mapsto (\log|x|/w, \mathrm{Arg}(x)/w) \in (\mathbb{R}^d/w, \mathbb{T}/w).$$

Note that if x is in a boundary toric divisor $Z_w, w \in \mathrm{vert}(T)$, then $\log|x|$ and $\mathrm{Arg}(x)$ are not well defined, but $\log|x|/w$ and $\mathrm{Arg}(x)/w$ are. Hence, the map ψ is well defined over $X(V_w^{\beta^\vee})$. The meaning of this map is the choice of local coordinates for Z_a near x (cf. [HZ02, Lemma 3.9]). Hence it is holomorphic.

Before defining the map over the charts U_v^β let us first make some estimates in the spirit of Lemma 4.1. For an element $\theta \in \mathbb{R}/\mathbb{Z}$ we will write $|\theta| < c$ if the (standard Euclidean) distance from θ to 0 is less than c. This inequality is vacuous for $c \ge 1/2$.

Lemma 4.2. *If a point x of Z_a lies in $X(U_v^\beta)$, then*

$$|\langle v, \log|x|\rangle + \lambda(v)| \le C(\beta)e^{-\beta}$$
$$|\langle v, \mathrm{Arg}(x)\rangle + \theta_v| \le C(\beta)e^{-\beta},$$

where $C(\beta) \to \log|\#\Delta_{\mathbb{Z}}|$ as $\beta \to \infty$.

Proof. First of all note that since $X(U_v^\beta) \cap Z_a \subset (\mathbb{C}\backslash\{0\})^d$, both $\log|x|$ and $\mathrm{Arg}(x)$ are well defined. Also by (2) of Lemma 3.2 the values of the vector field $\mathfrak{X}$ are in

$(\text{carrier}\, v)^\vee$. Hence, for any $q \in U_v^\beta$ the ray $\mathcal{F}_q$ is in $Q^\lambda_{(v|\{0\})}(\beta)$ and the standard estimates on values of the monomials at $x \in \log^{-1}(Q^\lambda_{(v|\{0\})}(\beta))$ apply:

$$|a_m x^m| \le e^{-\beta} |a_v x^v|, \text{ all } m \neq v, \{0\}.$$

Or putting them all together we have

$$\left| \frac{1}{a_v x^v} \sum_{m \neq v, \{0\}} a_m x^m \right| \le \#|\Delta_{\mathbb{Z}}| e^{-\beta}.$$

Hence,

$$\left| \log \left(1 + \frac{1}{a_v x^v} \sum_{m \neq v, \{0\}} a_m x^m \right) \right| \le C'(\beta) \cdot \#|\Delta_{\mathbb{Z}}| e^{-\beta},$$

where $C'(\beta) \to 1$ as $\beta \to \infty$. Writing the equation of Z_a in $X(U_v^\beta)$ as

$$a_v x^v \left(1 + \frac{1}{a_v x^v} \sum_{m \neq v, \{0\}} a_m x^m \right) = 1$$

or, equivalently,

$$\log(a_v x^v) = -\log \left(1 + \frac{1}{a_v x^v} \sum_{m \neq v, \{0\}} a_m x^m \right)$$

will give the claimed estimates. $\square$

Now, identifying the tangent spaces of the torus fibers $\mathbb{T}_x = \log^{-1}(x)$ with $\mathbb{R}^d$, we can pull back the vector field $\mathfrak{X}(x)$ to get a (constant) vector field on $\mathbb{T}_x$. Then the map ψ for the points in $X(U_v^\beta) \cap Z_a \subset (\mathbb{C} \backslash \{0\})^d$ will be defined as:

$$\begin{aligned} \psi(x) := \big(& \log(x) - (\langle v, \log(x) \rangle + \lambda(v)) \mathfrak{X}(\log |x|), \\ & \operatorname{Arg}(x) - (\langle v, \operatorname{Arg}(x) \rangle + \theta_v) \mathfrak{X}(\log |x|) \big) \in \big(\mathbb{R}^d_v(\lambda(v), \mathbb{T}_v(\theta_v) \big). \end{aligned}$$

Here if β is large enough, i.e. $C(\beta) e^{-\beta} < \frac{1}{2}$, then according to the Lemma 4.2 there is a preferred *continuous* lift of $\langle v, \operatorname{Arg}(x) \rangle + \theta_v$ to (the neighborhood of 0 in) $\mathbb{R}$. We use this lift to first define the value for $(\langle v, \operatorname{Arg}(x) \rangle + \theta_v)\mathfrak{X}$ in $\mathbb{R}^d$ and then project it back to $\mathbb{T} = \mathbb{R}^d / \mathbb{Z}^d$.

Since $\mathfrak{X}_q = w$ for $q \in V_w^{\beta\vee}$ the definition of the map ψ over the charts $V_w^{\beta\vee}$ is consistent with the above definition on possible overlaps $U_v^\beta \cap V_w^{\beta\vee}$. Thus, we have a well defined map $\psi : Z_a^{\text{sm}} \to W_a$ which is an embedding [HZ02].

4.2. **Estimates on complex structures and metrics.** Let J_{Z_a} and J_{W_a} denote the complex structure operators on the tangent spaces to Z_a and W_a respectively. We would like to say that the embedding ψ is holomorphic up to a small order terms.

We have already mentioned that ψ is *precisely* holomorphic over the charts $V_w^{\beta^\vee}$. To measure the discrepancy at $x \in U_v^\beta$ we will fix some (Euclidean) norm on $\mathbb{R}^{d-1}$ (they are all equivalent) to induce the norm on the tangent space $\mathbb{R}^{d-1} \oplus \mathbb{R}^{d-1} \cong T_x W_a$. Let $C_{\lambda,\nu}(\beta)$ be the (uniform on U_v^β) bound for the (λ, ν) bi-PIKAS metric $g_{\lambda,\nu}$ written in the affine coordinates in U_v^β.

Lemma 4.3. *As $\beta \to \infty, \beta^\vee \to 0$, the linear map*

$$d\psi \circ J_{Z_a} \circ (d\psi)^{-1} - J_{W_a} : T_x W_a \to T_x W_a$$

is of order $C_{\lambda-c,\nu}(\beta) \cdot O(\frac{1}{\beta^\vee} e^{-\beta})$.

Proof. First we apply estimates similar to those in Lemma 4.2 to the differential $d\psi$, which we think of as an element in $(\mathbb{R}^d \oplus \mathbb{R}^d) \otimes ((\mathbb{R}^d)^* \oplus (\mathbb{R}^d)^*)$.

$$d\psi = \begin{pmatrix} \mathbb{1} - \mathfrak{X} \otimes v - (\langle v, \log|x|\rangle + \lambda(v))\nabla\mathfrak{X} & 0 \\ -(\langle v, \operatorname{Arg}(x)\rangle + \theta_v)\nabla\mathfrak{X} & \mathbb{1} - \mathfrak{X} \otimes v \end{pmatrix} \begin{pmatrix} d\log|x| \\ d\operatorname{Arg}(x) \end{pmatrix}$$

Note that the complex structures J_{Z_a} and J_{W_a} would match exactly via $d\psi$ if there were no $\nabla\mathfrak{X}$ terms (this is what happens in the charts $V_w^{\beta^\vee}$ where $\mathfrak{X}$ is constant).

According to (2) of Lemma 3.2 $\langle v, \mathfrak{X}\rangle = 1$ in U_v^β. Hence the projection operator

$$\begin{pmatrix} \mathbb{1} - \mathfrak{X} \otimes v & 0 \\ 0 & \mathbb{1} - \mathfrak{X} \otimes v \end{pmatrix}$$

has a norm of order 1 when restricted to Z_a, and the desired bound on $d\psi \circ J_{Z_a} \circ (d\psi)^{-1} - J_{W_a}$ will follow from estimating the $\nabla\mathfrak{X}$ terms.

But according to the Lemma 4.2 we have the uniform bounds:

$$|\langle v, \log|x|\rangle + \lambda(v)| \leq Ce^{-\beta}, \ |\langle v, \operatorname{Arg}(x)\rangle + \theta_v| \leq Ce^{-\beta}.$$

On the other hand, by (3) of Lemma 3.2 the gradient $\nabla\mathfrak{X}(\log|x|)$ is bounded by $O(\frac{1}{\beta^\vee}) \cdot |g_{\lambda-c,\nu}(\log|x|)|$. □

The Kähler form on Z_a^{sm} is defined as the restriction of the Kähler form ω on X_T which in $(\mathbb{C}\backslash\{0\})^d$ is given by:

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Phi^{\mathrm{sm}}_{\lambda+\gamma,\nu,h}(\log|x|) + \epsilon\omega_{FS},$$

where ω_{FS} is a fixed (e.g., the Fubini-Study) Kähler form. We will compare the metric induced by ω with the (degenerate) scalar product on W_a induced by the (λ, ν)-bi-PIKAS.

To make these estimates we will need to introduce some bounds (in a Euclidean metric in $\mathbb{R}^d$) all of which follow essentially from the definition of the bi-PIKAS family:

$$
\begin{aligned}
&|g_{ij}(x)| < C_0(\beta), \text{ from Lemma 4.3 above, uniformly for } x \in U_v^\beta,\\
&|\mathrm{Hess}\,\Phi_{\lambda+\gamma}(x)| < C_1(\gamma), \text{ uniformly for } x \in \Delta^\vee_{\lambda+\gamma/2},\\
&|g_{\lambda+\gamma,\nu}(q) - g_{\lambda,\nu}(q)| < C_2(\gamma), \text{ uniformly for } q \in \partial\mathcal{U},\\
&\left|\mathrm{Hess}\,\Phi^{\mathrm{sm}}_{\lambda+\gamma,h}(x) - \mathrm{Hess}\,\Phi_{\lambda+\gamma}(x)\right| < C_3(h), \text{ uniformly for } x \in \Delta^\vee_{\lambda+\gamma/2},\\
&\left|\mathrm{Hess}\,\Phi_{\lambda+\gamma}|_{\mathbb{R}^d_v}(q + t\mathfrak{X}(x)) - \mathrm{Hess}\,\Phi_\lambda|_{\mathbb{R}^d_v}(q)\right| < C_4(\gamma), \text{ uniformly for}\\
&\qquad\qquad -\gamma < t < \gamma, q \in U_v^\beta \subset \partial\Delta^\vee_\lambda,\\
&|\mathrm{Hess}\,\Phi_\lambda(x) - \mathrm{Hess}\,K_w(q)| < C_5(\beta, c), \text{ uniformly for}\\
&\qquad\qquad q \in N^\beta(\partial\mathcal{U}) \cap V_w^{\beta^\vee}, x \in \mathcal{F}_q := q + \mathbb{R}_{\geq -c}\cdot w,
\end{aligned}
$$

where $C_0(\beta) \to \infty,\ C_1(\gamma) \to \infty,\ C_2(\gamma) \to 0,\ C_3(h) \to 0,\ C_4(t) \to 0,\ C_5(\beta, c) \to 0$ as (all of) the corresponding parameters go to 0. The last inequality follows from $\Phi_\lambda \in C^2(\partial\mathcal{U}\backslash D)$, where the local potential $K_w(x)$ is pulled back from the quotient.

Lemma 4.4. *Under the embedding $\psi : Z_a^{\mathrm{sm}} \to W_a$ the scalar products agree up to terms of order $C_2(\gamma) + C_3(h) + C_1(\gamma)\frac{C_0(\beta)}{\beta^\vee}e^{-\beta} + C_4(\gamma) + C_5(\beta, c) + O(\epsilon)$.*

Proof. Let $x \in X(U_v^\beta) \cap Z_a$. Assuming $e^{-\beta} < \gamma/2$ we have

$$\left|\mathrm{Hess}\,\Phi^{\mathrm{sm}}_{\lambda+\gamma,h}(x) - \mathrm{Hess}\,\Phi_{\lambda+\gamma}(x)\right| < C_3(h).$$

The difference between $\mathrm{Hess}\,\Phi_{\lambda+\gamma}|_{T_xZ_a}$ and $\mathrm{Hess}\,\Phi_{\lambda+\gamma}|_{T_{\psi(x)}W_a}$ consists of two terms. The first term $C_4(\gamma)$ appears from comparing $\mathrm{Hess}\,\Phi_{\lambda+\gamma}|_{\mathbb{R}^d_v}$ at $q + t\mathfrak{X}(x)$ with $\mathrm{Hess}\,\Phi_\lambda|_{\mathbb{R}^d_v}$ at q. The other term $C_1(\gamma)\frac{C_0(\beta)}{\beta^\vee}e^{-\beta}$ reflects the error in the alignment of the tangent spaces via the map $d\psi$ in the proof of Lemma 4.3.

Now let $x \in X(V_w^{\beta^\vee})$. We will just need to check points in $\mathcal{F}_q$ for $q \in V_w^{\beta^\vee} \setminus \bigcup U_v^\beta = N^\beta(\partial\mathcal{U}) \cap V_w^{\beta^\vee}$. Note that $\mathrm{Hess}\,\Phi_\lambda$ is bounded in $\mathcal{F}(V_w^{\beta^\vee})$ and is continuous at $\partial\mathcal{U}\backslash D$ (in particular, the Hessian vanishes into the w-direction at $\partial\mathcal{U}\backslash D$). Hence the C_5 bound from above are also valid for the regularization $\mathrm{Hess}\,\Phi^{\mathrm{sm}}_{\lambda+\gamma,h}$. Namely,

$$\left|\mathrm{Hess}\,\Phi^{\mathrm{sm}}_{\lambda+\gamma,h}(x) - g_{\lambda+\gamma,\nu}(q)\right| < C_5(\beta, c),$$

with possibly different function C_5. The discrepancy between $g_{\lambda+\gamma,\nu}$ and $g_{\lambda,\nu}$ in $N^\beta(\partial\mathcal{U}) \cap V_w^{\beta^\vee}$ is encoded in the $C_2(\gamma)$ term.

Finally, the term $\epsilon\omega_{0,\nu}$ in ω can be bounded by $O(\epsilon)$. □

4.3. **The Gromov-Hausdorff limits of one-parameter families.** We will apply the results of the previous sections to the situation considered in [HZ02] to draw a consequence mostly related to the mirror symmetry conjecture. Let λ_0 be an integral vector in the interior of the secondary cone $\mathrm{SC}(S)$. We consider an 1-parameter family of the hypersurfaces Z_s defined as closures in X_T of

$$Z_s^{\mathrm{aff}} := \{x \in (\mathbb{C}\backslash\{0\})^d \ : \ \sum_{m\in\Delta_{\mathbb{Z}}\backslash\{0\}} a_m s^{\lambda_0(m)} x^m = 1\}.$$

Choose an integral vector ν_0 in the interior of $\mathrm{SC}(T)$ and consider Σ with the metric space structure given by the bi-PIKAS (λ_0, ν_0).

Also consider an one-parameter family of (non-compact) Kähler manifolds $W_s := W(\log|a| + \log|s|\cdot\lambda_0, \mathrm{Arg}(a) + \mathrm{Arg}(s)\cdot\lambda_0)$, whose metric and complex structure are induced from the $\left(\log|a| + \log|s|\cdot\lambda_0, \frac{\nu_0}{\log|s|}\right)$ bi-PIKAS.

Theorem 4.5. *As $|s|\to\infty$ one can choose smooth portions of the hypersurfaces $Z_s^{\mathrm{sm}} \subset Z_s$, the embeddings $\psi_s : Z_s^{\mathrm{sm}} \hookrightarrow W_s$ and a family of Kähler metrics on Z_s in the class $\frac{\nu_0}{\log|s|}(1+o(1))$ such that the pairs $(Z_s, Z_s\backslash Z_s^{\mathrm{sm}})$ converges to the pair (Σ, D) in the Gromov-Hausdorff sense, and the maps ψ_s identify (uniformly in $x \in Z_s^{\mathrm{sm}}$) the scalar products and the complex structures on the tangent spaces T_xZ_s and $T_{\psi_s(x)}W_s$ up to terms of order $o(1)$.*

Proof. We consider the bi-PIKAS family in a neighborhood of (λ_0, ν_0) and extend it by rescaling to (a neighborhood of) the ray $(\lambda, \nu) = \left(\log|a| + \log|s|\cdot\lambda_0, \frac{\nu_0}{\log|s|}\right)$ in $\mathrm{SC}(S)\times\mathrm{SC}(T)$.

The C_i-estimates for the (λ, ν) bi-PIKAS considered in the bi-PIKAS metric (rather than in Euclidean) are equivalent to the corresponding estimates for the rescaled structure $((\log|s|)^{-1}\lambda, \log|s|\nu)$ made in the Euclidean metric as before. This is because the Euclidean metric on $\partial\Delta_{\lambda_0}$ is equivalent to the bi-PIKAS metric on Σ. But to pass from (λ, ν) to $((\log|s|)^{-1}\lambda, \log|s|\nu)$ we will need to rescale all the parameters as well:

$$\lambda \sim \lambda_0 \log|s|,\ \beta \sim \beta_0 \log|s|,\ \gamma \sim \gamma_0 \log|s|,\ h \sim h_0 \log|s|,\ c \sim c_0 \log|s|,$$
$$\nu \sim \frac{\nu_0}{\log|s|},\ \beta^\vee \sim \frac{\beta_0^\vee}{\log|s|},\ \mathrm{Hess}\,\Phi \sim \frac{\mathrm{Hess}\,\Phi_0}{(\log|s|)^2}.$$

Or equivalently, we could apply the log map with the base $|s|$ as in [HZ02].

We saw in the proof of Lemma 4.4 that $\mathrm{Hess}\,\Phi^{\mathrm{sm}}_{\lambda_0+\gamma_0}$ is degenerate along $\mathcal{F}$ outside $\Delta^\vee_{\lambda_0+\gamma_0+h_0}$. This argument extended to the entire toric variety shows that up to terms of order $O(\epsilon)$, the set $X_T\backslash \log^{-1}(\Delta^\vee_{\lambda_0+\gamma_0+h_0})$ (which contains Z_s) has distance from $\partial\Delta^\vee_\lambda$ bounded by the diameters of the torus fibers $\mathbb{T}$. The size of the tori $\mathbb{T}$ is determined by the norm of $\mathrm{Hess}\,\Phi^{\mathrm{sm}}_{\lambda+\gamma}$ at the corresponding point, which is bounded by $\frac{C_1(\gamma_0)}{(\log|s|)^2}$.

The rest of the proof consists of careful picks for asymptotics of the rescaled parameters $\beta_0, \beta_0^\vee, h, \gamma_0, c_0, \epsilon$ to ensure that the following expressions

$$\beta_0, \quad \beta_0^\vee, \quad \gamma_0, \quad c_0, \quad h_0,$$
$$C_1(\gamma_0)(\log|s|)^{-2},$$
$$\log|s|\frac{1}{\beta_0^\vee}e^{-\beta_0\log|s|}\cdot C_0(\beta_0),$$
$$\log|s|\frac{C_0(\beta_0)C_1(\gamma_0)}{\beta_0^\vee}e^{-\beta_0\log|s|} + C_2(\gamma_0) + C_3(h_0)$$
$$+C_4(\gamma_0) + C_5(\beta_0, c_0) + O(\epsilon)(\log|s|)^2$$

go to 0 as $\log|s| \to \infty$, where the first two lines take care of the Hausdorff convergence, and the last two give matching of the complex structure and the metric under the embedding $\psi : Z_s^{\mathrm{sm}} \to W_s$

For instance, we can choose

$$\beta_0^\vee \sim \frac{1}{\log|s|}, \quad \gamma_0 \sim C_1^{-1}(\log|s|), \quad c_0 \sim \frac{1}{\log|s|}, \quad h_0 \sim \frac{1}{\log|s|}.$$

And $\beta_0(\log|s|)$ has satisfy $C_1(\frac{1}{\log|s|}e^{-\beta_0\log|s|}) < \log|s|$ (i.e. $e^{-\beta} < \gamma$, which is needed for the proof of Lemma 4.4) and

$$(\log|s|)^3 C_0(\beta_0)e^{-\beta_0\log|s|} \to 0,$$

which is possible due to the fast decreasing factor of $e^{-\beta_0\log|s|}$ when $\beta_0(\log|s|)$ is changing slowly.

□

Remark. We can rephrase the above theorem in terms of the alternate definition of the torus bundles $W_{\frac{1}{\log|s|}}(\lambda_0, \nu_0)$ associated to the given Kähler affine structure on Σ. Then the statement of the theorem will coincide with the Conjecture 2 of [KS01].

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