DUKE-CGTP-05-03

INTEGRAL AFFINE STRUCTURES ON SPHERES III: COMPLETE INTERSECTIONS

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Abstract. We give a construction of integral affine structures on spheres which conjecturally arise in the maximal degeneration limits of Calabi-Yau complete intersection families in toric varieties. This generalizes our model for affine structures on toric Calabi-Yau hypersurfaces [HZ02].

1. Introduction

Starting with the following data we will construct a pair of affine structures on a sphere (or a product of spheres) with a codimension 2 discriminant locus and whose monodromy representations have dual linear parts. Let $\Delta = \Delta^{(1)} + \cdots + \Delta^{(r)} \subset (\mathbb{R}^d)^*$ be a nef-partition of a d-dimensional *reflexive* polytope Δ. That is for each $i = 1, \ldots, r$ the polytope $\Delta^{(i)}$ contains the origin and there is a *convex integral* PL function ψ_i which has values 1 on every nonzero vertex of $\Delta^{(i)}$ and 0 on the other polytopes $\Delta^{(j)}$.

Let $\nabla = \nabla^{(1)} + \cdots + \nabla^{(r)} \subset \mathbb{R}^d$ be the dual nef-partition (also d-dimensional) as in [BB96]. Explicitly, if ϕ_i is the Legendre dual to the zero function on $\Delta^{(i)}$, i.e.,

$$\phi_i(x) = \max_{y \in \Delta^{(i)}} \{\langle y, x \rangle\},$$

then the polytopes $\nabla^{(i)}$ are defined as:

$$\nabla^{(i)} := \operatorname{conv}\{0,\ x \in \Delta^\vee \ \ : \ \phi_i(x) = 1\}.$$

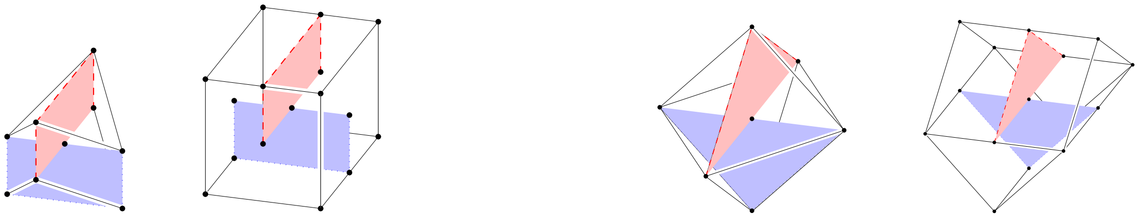
(Here and later we will always denote by $P^\vee$ the polytope dual to P.) In particular, the vertices of $\Delta^{(i)}$ are the gradients of the piece-wise linear function ϕ_i, and the vertices of $\nabla^{(j)}$ are the gradients of the function ψ_j. Then we have the duality (cf. [Bor93]):

$$\nabla^\vee = \operatorname{conv}\{\Delta^{(1)}, \ldots, \Delta^{(r)}\} \text{ and } \Delta^\vee = \operatorname{conv}\{\nabla^{(1)}, \ldots, \nabla^{(r)}\}.$$

The other part of our input are two functions defined on the lattice points of $\nabla^\vee$ and $\Delta^\vee$:

$$\omega : \nabla^\vee_{\mathbb{Z}} \to \mathbb{R}, \qquad \nu : \Delta^\vee_{\mathbb{Z}} \to \mathbb{R},$$

The first author gratefully acknowledges support by NSF-grant DMS-0200740. The second author was partially supported by NSF grant DMS-0405939.



$\nabla^\vee = \mathrm{conv}(\Delta^{(1)}, \Delta^{(2)})$ $\quad \Delta = \Delta^{(1)} + \Delta^{(2)}$ $\quad \Delta^\vee = \mathrm{conv}(\nabla^{(1)}, \nabla^{(2)})$ $\quad \nabla = \nabla^{(1)} + \nabla^{(2)}$

FIGURE 1. Dual nef-partitions.

such that the induced subdivisions of $\nabla^\vee$ and $\Delta^\vee$ are central, i.e. every maximal cell contains the origin. We will denote by S and T the induced subdivisions of the boundaries of $\nabla^\vee$ and $\Delta^\vee$ respectively. To simplify the notation, we will add constants to ω and ν so that $\omega(0) = \nu(0) = 0$.

If ω and ν are integer valued then to such a data one can associate a one-parameter family of complete intersections in a toric variety by taking the closure of the affine complete intersection defined by r Laurent polynomials in $(\mathbb{C}^*)^d$:

$$f_i(z) = \sum_{m \in (\Delta^{(i)})_{\mathbb{Z}}} \lambda^{\omega(m)} z^m$$

in the toric variety X_T defined by the fan over the subdivision T. The function ν gives an integral Kähler class on X_T which restricts to a class on the complete intersection. The constructed affine structures are conjectured to be the ones which arise in the metric collapse of the corresponding family.

We do not discuss the geometry of degenerations here (see a recent article by Mark Gross [Gro04] for more details on that aspect). In the present paper we clarify the combinatorics of the model. In particular, we establish a homeomorphism between the model and a sphere. The affine structure and monodromy calculations are in agreement with those in [Gro04].

Acknowledgments. We are indebted to Lev Borisov who provided several key ideas. We are also very grateful to Mark Gross for communicating to us Example 3.4 which showed that the affine structure can extended beyond the naïve bipartite cover.

Sphericity of our model was discussed with Richard Ehrenborg and Margaret Readdy, and with Vic Reiner. Frank Lutz, Niko Witte and Günter Ziegler came up with a counterexample to an overly optimistic conjecture of the first author.

2. THE MODEL

2.1. **Semi-simplicity of nef-partitions.** A nef-partition $\Delta = \Delta^{(1)} + \cdots + \Delta^{(r)}$ is called *reducible* (cf. [BB96]) if there exists a proper subset $\{i_1, \ldots, i_s\}$ of the set

$[r] = \{1, \dots, r\}$ such that the polytope $\Delta^{(i_1)} + \cdots + \Delta^{(i_s)}$ contains 0 in its relative interior. Otherwise the nef-partition is *irreducible*.

Theorem 2.1 ([BB96])**.** *Any nef-partition is a direct sum of irreducibles.*

Remark. In this decomposition one may need to refine the sum lattice so that it contains the direct sum of constituent lattices as a finite index sublattice.

From now on we will restrict our attention to irreducible nef-partitions. Only those will correspond to true Calabi-Yau families. Direct sums will correspond to the products of Calabi-Yau complete intersections of smaller dimensions, up to a finite group action (cf. [BB96]).

2.2. **The sphere.** Consider the product $\Delta \times \nabla \subset (\mathbb{R}^d)^* \times \mathbb{R}^d$. The complex Σ — our prospective sphere — will be a subdivision of

$$|\Sigma| = \{(m,n) \in \Delta \times \nabla \ : \ \langle m,n \rangle = r\}. \tag{1}$$

The rest of this subsection will be devoted to defining this subdivision.

Given $\sigma \in S$ we set $\sigma^{(i)} := \sigma \cap \Delta^{(i)}$, and more generally $\sigma^I := \mathrm{conv}(\sigma^{(i)} \ : \ i \in I)$ for a set $I \subseteq [r]$ of indices. Also denote $I_\sigma := \{i \ : \ \sigma^{(i)} \neq \emptyset\}$.

We will say that σ is *transversal* if $I_\sigma = [r]$, that is, if $\sigma^{(i)}$ is not empty for every $i = 1, \dots, r$. The transversal cells form an upper order ideal P in the face lattice of S. For $\sigma \in P$ we will consider the subset σ_Δ of Δ:

$$\sigma_\Delta := \sigma^{(1)} + \cdots + \sigma^{(r)}.$$

(For non-transversal σ this yields the empty set.) We denote the collection $\{\sigma_\Delta \ : \ \sigma \in P\}$ by S_Δ.

Proposition 2.2. *The collection S_Δ forms a polytopal complex whose face lattice is isomorphic to the poset P. In particular, the vertices of S_Δ are σ_Δ for minimal $\sigma \in P$.*

The space $|S_\Delta| \subset \partial\Delta$ coincides with the image of the natural projection map $p_1 : |\Sigma| \to \Delta$.

Proof. Here is an alternative definition of S_Δ from which the first assertion is obvious. The space $|S_\Delta|$ is the intersection of Δ with the boundary of $r\nabla^\vee$. The cells of the complex S_Δ are $\sigma_\Delta = r\sigma \cap \Delta$.

For the second assertion, if $x = p_1(x,y)$ with $(x,y) \in |\Sigma|$, then $\langle \frac{1}{r}x, y \rangle = 1$ shows that $x \in \partial(r\nabla^\vee)$. So $x \in |S_\Delta|$.

Conversely, if $x \in \sigma_\Delta$, let $y \in \Delta^\vee$ with $\langle \sigma, y \rangle = 1$. Then $(x,y) \in |\Sigma|$. □

Lemma 2.3. *For every (not necessarily transversal) $\sigma \in S$ and any index set $I \subseteq [r]$, σ^I is a face of σ. If the two faces σ^I and $\sigma^{\bar{I}}$ are non-empty, then the lattice distance between them is one. In particular, minimal transversal cells σ are unimodular $(r-1)$-simplices.*

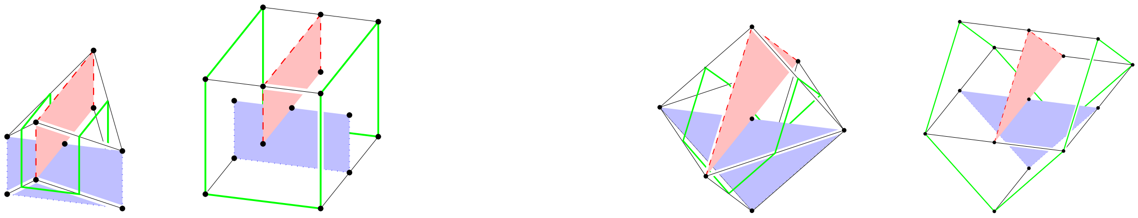


FIGURE 2. $|S_\Delta| = \partial(r\nabla^\vee) \cap \Delta$.

Proof. The integral PL function $\sum_{i\in I} \phi_i$ is linear on σ, equals 1 on σ^I and vanishes on $\sigma^{\bar{I}}$.

If σ is a minimal transversal cell, then σ_Δ is a vertex so that each $\sigma^{(i)}$ is a vertex. Thus, σ is a simplex which has facet width one by the first assertion. □

We can repeat everything said above for the dual data $(\nabla, \nabla^{(j)}, T, \tau^{(j)})$ instead of $(\Delta, \Delta^{(i)}, S, \sigma^{(i)})$. In particular, the poset Q will be the collection of all transversal cells $\tau \in T$. The subdivision $T_\nabla := \{\tau_\nabla \ : \ \tau \in Q\}$ of the relevant part of the boundary of ∇ will coincide with the image of the projection $p_2 : |\Sigma| \to \nabla$.

We say that a pair $(\sigma, \tau) \in P \times Q$ is *adjoint* if $\langle \sigma_\Delta, \tau_\nabla \rangle = r$. Since $\langle \Delta^{(i)}, \nabla^{(j)} \rangle \le \delta_{ij}$ a pair (σ, τ) is adjoint if and only if $\langle \sigma^{(i)}, \tau^{(j)} \rangle = \delta_{ij}$.

Definition. The complex Σ consists of the cells $\sigma_\Delta \times \tau_\nabla$ for all adjoint pairs (σ, τ).

Remark. Everything said above is valid for reducible nef-partitions as well. If $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_k$ is a direct sum of k irreducibles, then $\Sigma = \Sigma_1 \times \cdots \times \Sigma_k$. Later we will prove that for an irreducible nef-partition Σ is homeomorphic to the $(d-r)$-dimensional sphere; for reducible ones it is a product of spheres.

2.3. **Tropical amoebas.** The function $\omega : \nabla^\vee_{\mathbb{Z}} \to \mathbb{R}$ and its restrictions $\omega_i : \Delta^{(i)}_{\mathbb{Z}} \to \mathbb{R}$ define tropical amoebas $\mathcal{A}$ and $\mathcal{A}^{(i)}$ as follows. Consider the ω-lifted polytopes in $(\mathbb{R}^d)^* \times \mathbb{R}$

$$\tilde{\nabla}^\vee = \operatorname{conv}\left[\begin{pmatrix} m \\ \omega(m) \end{pmatrix} \ : \ m \in \nabla^\vee_{\mathbb{Z}}\right] \text{ and } \tilde{\Delta}^{(i)} = \operatorname{conv}\left[\begin{pmatrix} m \\ \omega(m) \end{pmatrix} \ : \ m \in \Delta^{(i)}_{\mathbb{Z}}\right].$$

The lower convex hull of $\tilde{\nabla}^\vee$ projects to the subdivision $S \star 0$ of $\nabla^\vee$. The normal fan of $\tilde{\nabla}^\vee$ subdivides $\mathbb{R}^d \cong \mathbb{R}^d \times \{-1\}$ into cells

$$\begin{aligned} F_\sigma &= \mathrm{NC}_{\tilde{\nabla}^\vee}(\tilde{\sigma}) \ \cap \ (\mathbb{R}^d \times \{-1\}) \\ &= \{y \ : \ \langle m, y\rangle - \omega(m) = \max_{m' \in \nabla^\vee_{\mathbb{Z}}} \langle m', y\rangle - \omega(m') \text{ for all vertices } m \text{ of } \sigma\}. \end{aligned}$$

Similarly, for $\sigma \subseteq \Delta^{(i)}$,

$$\begin{aligned} F^{(i)}_\sigma &= \mathrm{NC}_{\tilde{\Delta}^{(i)}}(\tilde{\sigma}) \ \cap \ (\mathbb{R}^d \times \{-1\}) \\ &= \{y \ : \ \langle m, y\rangle - \omega(m) = \max_{m' \in \Delta^{(i)}_{\mathbb{Z}}} \langle m', y\rangle - \omega(m') \text{ for all vertices } m \text{ of } \sigma\}. \end{aligned}$$

The amoebas $\mathcal{A}$ and $\mathcal{A}^{(i)}$ are the polyhedral subcomplexes of the cells of positive codimension:

$$\mathcal{A} := \bigcup \{F_\sigma \ : \ \sigma \in S \star 0, \dim \sigma > 0\}$$
$$\mathcal{A}^{(i)} := \bigcup \{F_\sigma^{(i)} \ : \ \sigma \in \Delta^{(i)}, \sigma \in S \star 0, \dim \sigma > 0\}$$

We will be interested in the subcomplex ∇_ω of $\mathcal{A}$ which consists of all cells $F_{\bar\sigma}$ for transversal $\sigma \in P$, where $\bar\sigma = \sigma \star 0$. Observe that with this definition, the barycentric subdivision $\operatorname{bsd}(\nabla_\omega)$ is isomorphic to the order complex of P, i.e., the subcomplex of $\operatorname{bsd}(S)$ induced by barycenters of transversal cells.

Proposition 2.4. *∇_ω is equal to the complex of bounded cells of $\bigcap \mathcal{A}^{(i)}$.*

Proof. We will show that the complexes of bounded cells of $\mathcal{A}$ and of $\bigcup \mathcal{A}^{(i)}$ agree with the boundary of the polytope $F_0 = \bigcap F_0^{(i)}$.

As the normal fan of the Minkowski sum $\tilde\Delta = \sum \tilde\Delta^{(i)}$ is the common refinement of normal fans of the $\tilde\Delta^{(i)}$, the subdivision of $\mathbb{R}^d \cong \mathbb{R}^d \times \{-1\}$ by the normal fan of $\tilde\Delta$ is the subdivision by $\bigcup \mathcal{A}^{(i)}$.

The polytope $\tilde\nabla^\vee$ generates a cone C with apex $\binom{0}{0}$. Its proper faces project to the cones over the cells of S. The lower convex hull of each of the $\tilde\Delta^{(i)}$ is contained in the boundary of C.

The Minkowski sum $\tilde\Delta = \sum \tilde\Delta^{(i)}$ generates the same cone: on one hand $\tilde\Delta \supseteq \tilde\nabla^\vee$, on the other hand the Minkowski sum of subsets of C is contained in C. Moreover, the lower convex hull of $\tilde\Delta$ is contained in the boundary of C as well: the sums $\sum \tilde\sigma^{(i)} \subseteq \partial C$ for $\sigma \in S \star 0$ belong to the lower hull, and their projections, the $\sum \sigma^{(i)}$, cover Δ.

The lower convex hull of $\tilde\Delta$ projects to a mixed subdivision of $\Delta = \sum \Delta^{(i)}$. By the above observations, this is the subdivision of Δ by the cones over cells of S. That is, the cells are precisely the $\sum \sigma^{(i)}$ where σ runs over the cells of $S \star 0$.

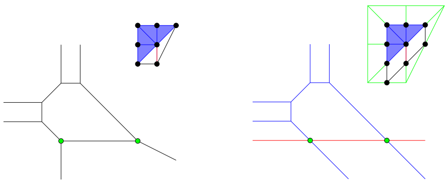


FIGURE 3. $\mathcal{A}$ versus $\bigcup \mathcal{A}^{(i)}$.

Therefore, the bounded cells of both subdivisions are the intersection of the normal fan of C with $\mathbb{R}^d \times \{-1\}$, and we have $F_{\bar\sigma} = \bigcap F^{(i)}_{\bar\sigma^{(i)}}$ for $\sigma \in S$. □

Theorem 2.5. *∇_ω is homeomorphic to the $(d-r)$-sphere.*

We subdivide the proof into several lemmas which might be of independent interest.

Lemma 2.6. *There are vectors $v_i \in \operatorname{relint} \Delta^{(i)}$ so that $\sum v_i = 0$, and similarly $w_i \in \nabla^{(i)}$ so that $\sum w_i = 0$. The v_i and the w_i positively span $(r-1)$-dimensional linear spaces V and W respectively, and $(\mathbb{R}^d)^* = V \oplus W^\perp$.*

Proof. We can write the origin as a convex combination (that is, a linear combination whose coefficients sum to 1) of the vertices of $\nabla^\vee$ with all coefficients *positive*. Define v_i to be the contribution of the vertices of $\Delta^{(i)}$ to this sum.

If we denote $w^I := \sum_{i \in I} w_i$ for index sets $I \subseteq [r]$, then we see that $w^{\bar{I}}$ is non-positive on $\Delta^{(i)}$ for $i \in I$ so that $w^I = -w^{\bar{I}}$ is non-negative on $\Delta^{(i)}$ for $i \in I$.

Because the nef-partition is irreducibile, no proper subcollection of the v_i or of the w_i contains zero in their convex hull. Hence we have $(-1)^{\delta_{ij}} \langle v_i, w_j \rangle < 0$, and any proper subcollection of the v_i spans a pointed cone. The remaining assertions follow. $\square$

Now shift the $\Delta^{(i)}$: set $\Delta_s^{(i)} = \Delta^{(i)} + v_i$, and for $\sigma \in S$ set $\sigma_s = \operatorname{conv}(\sigma^{(i)} + v_i)$.

Lemma 2.7. *The σ_s subdivide the boundary of $\nabla_s^\vee = \operatorname{conv}(\Delta_s^{(1)}, \ldots, \Delta_s^{(r)})$ with the same combinatorics as S.*

Proof. We need to show that σ_s belongs to the boundary of $\nabla_s^\vee$. The integral PL functions ϕ_j restrict to linear functions y_j on σ. We have $y_j(\sigma^{(i)}) = \delta_{ij}$, and $y_j(\Delta^{(i)}) \leq \delta_{ij}$. Thus $y = \sum y_j$ defines a face of $\nabla^\vee$ containing σ.

Furthermore, the $\lambda_{ij} := \langle y_j, v_i \rangle$ satisfy $(-1)^{\delta_{ij}} \lambda_{ij} < 0$, and $\sum_i \lambda_{ij} = 0$. Thus, the $r \times r$ matrix $\Lambda + \mathrm{id} = (\lambda_{ij} + \delta_{ij})_{1 \leq i,j \leq r}$ is invertible and we can find coefficients α_j so that for all i, $\alpha_i + \sum_j \lambda_{ij} \alpha_j = 1$. We claim that $\alpha_j > 0$.

Assume $\alpha_j > 0$ for $j \in J_+$, $\alpha_j = 0$ for $j \in J_0$, and $\alpha_j < 0$ for $j \in J_-$. We have

$$\sum_j \alpha_j = \mathbb{1}^t \alpha = \mathbb{1}^t (\Lambda + \mathrm{id}) \alpha = \mathbb{1}^t \mathbb{1} = r,$$

where $\mathbb{1}$ is the all-one-vector. So $J_+ \neq \emptyset$. Consider the (row) vector β with $\beta_j = |J_-|$ for $j \in J_+$, $\beta_j = 0$ for $j \in J_0$, and $\beta_j = -|J_+|$ for $j \in J_-$. Then

$$\beta(\Lambda + \mathrm{id})\alpha = \sum_j \beta_j = 0.$$

On the other hand, for $j \in J_+$, the j^{th} component of $\beta(\Lambda + \mathrm{id})$ is

$$\underbrace{|J_-| \left(\lambda_{jj} + \sum_{i \in J_+ \setminus j} \lambda_{ij} \right)}_{\geq 0} - |J_+| \underbrace{\sum_{i \in J_-} \lambda_{ij}}_{\leq 0} + |J_-| > 0 \text{ unless } J_- = \emptyset.$$

Similarly, for $j \in J_-$, the j^{th} component of $\beta(\Lambda + \mathrm{id})$ is

$$|J_-| \underbrace{\sum_{i\in J_+} \lambda_{ij}}_{\le 0} - |J_+| \underbrace{\left(\lambda_{jj} + \sum_{i\in J_-\setminus j} \lambda_{ij}\right)}_{\ge 0} - |J_+| < 0 \text{ as } J_+ \neq \emptyset.$$

From these equations we deduce the contradiction $\beta(\Lambda + \mathrm{id})\alpha > 0$ unless $J_- = \emptyset$, that is $\alpha \geq 0$.

We define $y_s := \sum \alpha_i y_i$ to get $\langle y_s, \sigma_s^{(i)}\rangle = 1$, and $\langle y_s, \Delta_s^{(i)}\rangle \leq 1$. □

Proof of Theorem 2.5. The face lattice of ∇_ω is opposite isomorphic to the lattice P of transversal cells of S.

We claim that $W^\perp$ intersects precisely the (shifted) transversal cells so that the induced subdivision of the boundary of the convex $(d-r+1)$-polytope $\nabla_s^\vee \cap W^\perp$ has face lattice P.

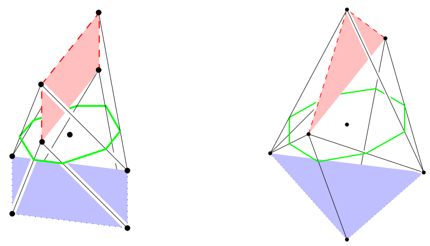


FIGURE 4. $\nabla_s^\vee \cap W^\perp$ and $\Delta_s^\vee \cap V^\perp$.

- Suppose $\sigma \in S$ is not transversal, that is $I_\sigma \subsetneq [r]$. Then σ_s can be separated from $W^\perp$ by w^{I_σ}.
- Conversely, suppose σ_s can be separated from $W^\perp$ by some linear functional w. That is, $w \in (W^\perp)^\perp$ and $w > 0$ on σ_s. Then w can be written as a positive combination of a proper subset of the w_i. But if w_j is not in this collection, then $w < 0$ on $\sigma_s^{(j)}$. Therefore $\sigma_s^{(j)}$ is empty, and σ was not transversal.

□

Corollary 2.8. *If $\Delta = \Delta^{(1)} + \cdots + \Delta^{(r)}$ is an irreducible nef-partition, then the topological space Σ is homeomorphic to the sphere S^{d-r}.*

Proof. There is a (pulling) subdivision of F_0 that restricts to a subdivision of ∇_ω isomorphic to the barycentric subdivision $\mathrm{bsd}(\Sigma)$. □

3. Affine structure on Σ

3.1. **The smooth part and the discriminant.** The combinatorial structure of the affine (smooth) part of Σ is more complex in the complete intersection case than it is for hypersurfaces. The main difference is that the affine structure can be extended beyond the bipartite covering. In particular, the nerve of this enlarged covering is not a graph and the fundamental group is not free.

For every adjoint pair (σ,τ) we take the open star neighborhood in $\operatorname{bsd}\Sigma$:

$$U_{(\sigma,\tau)} := \operatorname{Star}(\hat\sigma_\Delta \times \hat\tau_\nabla).$$

The collection $\{U_{(\sigma,\tau)}\}$ provides an open covering of Σ.

For each minimal transversal $\sigma \in P$ it is convenient to combine all charts $\operatorname{Star}(\hat\sigma_\Delta \times \hat\tau_\nabla)$ into a single chart U_σ. Analogously, $V_\tau := \bigcup \operatorname{Star}(\hat\sigma_\Delta \times \hat\tau_\nabla)$ for a fixed minimal transversal $\tau \in Q$. In terms of the two projections

$$p_1\colon\ \operatorname{bsd}(\Sigma) \to \operatorname{bsd}(S_\Delta), \quad p_2\colon\ \operatorname{bsd}(\Sigma) \to \operatorname{bsd}(T_\nabla),$$

U_σ and V_τ are the preimages of the open star neighborhoods of the vertices $\sigma_\Delta = \hat\sigma_\Delta$ and $\tau_\nabla = \hat\tau_\nabla$ in the barycentric subdivision of S_Δ and T_∇ respectively:

$$U_\sigma = p_1^{-1}(\operatorname{Star}(\sigma_\Delta)), \qquad V_\tau = p_2^{-1}(\operatorname{Star}(\tau_\nabla)).$$

Then the closures of the U_σ's cover the whole $|\Sigma|$ and the combinatorial structure of this covering is dual to the poset P. Similarly, $\{\bar V_\tau\}$ is a covering of $|\Sigma|$ dual to the poset Q.

We say that that an adjoint pair (σ,τ) is *smooth* if

$$\dim \sigma^{(i)} \cdot \dim \tau^{(i)} = 0, \qquad \text{for all } \ i = 1,\ldots,r.$$

Obviously, a pair (σ,τ) is smooth if either σ or τ is minimal. The *discriminant locus* D is the full subcomplex of $\operatorname{bsd}(\Sigma)$ generated by the vertices $(\hat\sigma_\Delta \times \hat\tau_\nabla)$ for all *non-smooth* adjoint pairs (σ,τ). Next we will describe an affine structure on $\Sigma \setminus D$.

First, we fix a homeomorphism $\phi : \Sigma \to \nabla_\omega$, which sends U_σ to the corresponding (maximal) cell $F_{\bar\sigma}$ of ∇_ω. (Recall that the $\bar\sigma = \sigma \star 0$ for transversal σ parameterize the cells of ∇_ω.) For a smooth pair (σ,τ) we choose a partition $\{I,J\}$ of the set $[r]$ such that $\dim\sigma^{(i)} = 0$ for all $i \in I$ and $\dim\tau^{(i)} = 0$ for $j \in J$.

Note that $\phi(U_{(\sigma,\tau)}) \subset \operatorname{aff} F_{\bar\sigma^I} = \{x \in \mathbb{R}^d \ : \ \langle m,x\rangle - \omega(m) = 0,\ m \in \sigma^I\}$. Then the integral affine structure on $U_{(\sigma,\tau)}$ is induced via ϕ by taking the quotient of a subspace in $\mathbb{R}^d$:

$$\operatorname{aff} F_{\bar\sigma^I}/\tau^J.$$

Any other allowable partition $\{I',J'\}$ will give a *canonically* equivalent structure. The easiest way to see this is to choose the partition $\{I'',J''\} := \{I\cap I', J\cup J'\}$:

$$\operatorname{aff} F_{\bar\sigma^I}/\tau^J \to \operatorname{aff} F_{\bar\sigma^{I''}}/\tau^{J''}$$
$$\operatorname{aff} F_{\bar\sigma^{I'}}/\tau^{J'} \to \operatorname{aff} F_{\bar\sigma^{I''}}/\tau^{J''}$$

The condition $\langle\sigma^{(i)},\tau^{(i)}\rangle = 1$ ensures that the above maps are isomorphisms of *integral* affine spaces.

In particular, the integral affine structure on U_σ is induced via the inclusion

$$\operatorname{aff} F_{\bar\sigma} \subset \mathbb{R}^d$$

from the standard integral affine structure on $\mathbb{R}^d$. The integral affine structure on V_τ comes from the quotient $\mathbb{R}^d/\tau$. The identification of the affine coordinates on the overlaps is provided via the quotient map.

3.2. **Monodromy and extention of the affine structure.** The nerve of the partial covering $\{U_\sigma, V_\tau\}$ is the bipartite graph Γ whose nodes are labeled by the corresponding minimal transversal cells $\sigma \in P$ and $\tau \in Q$. We call a loop in Γ *primary* if it has exactly 4 nodes $\sigma_0, \tau_0, \sigma_1, \tau_1$ so that $\operatorname{conv}(\sigma_0, \sigma_1) \in S$ and $\operatorname{conv}(\tau_0, \tau_1) \in T$. We denote such a loop by $(\sigma_0\tau_0\sigma_1\tau_1)$, and think of it as an element (possibly trivial) of the fundamental group $\pi_1(\Sigma \setminus D)$ with a base point in U_{σ_0}. The primary loops generate $\pi_1(\Gamma)$, and hence $\pi_1(\Sigma \setminus D)$, but there are many relations.

Proposition 3.1. *The monodromy transformation* $T_{(\sigma_0\tau_0\sigma_1\tau_1)}\colon \operatorname{aff} F_{\bar\sigma_0} \to \operatorname{aff} F_{\bar\sigma_0}$ *along the loop* $(\sigma_0\tau_0\sigma_1\tau_1)$ *is given by*

$$T_{(\sigma_0\tau_0\sigma_1\tau_1)}(x) = x + \sum_{j=1}^{r} \left[\langle \sigma_1^{(j)}, x\rangle - \omega(\sigma_1^{(j)})\right] (\tau_1^{(j)} - \tau_0^{(j)}).$$

Proof. The vectors $\sigma_0^{(i)}, \sigma_1^{(i)} \in (\mathbb{R}^d)^*$ and $\tau_0^{(j)}, \tau_1^{(j)} \in \mathbb{R}^d$ satisfy $\langle \sigma_{0,1}^{(i)}, \tau_{0,1}^{(j)}\rangle = \delta_{ij}$. Hence, if $x \in \operatorname{aff} F_{\bar\sigma_0}$, then

$$x + \sum_{j=1}^{r} \left[\langle \sigma_1^{(j)}, x\rangle - \omega(\sigma_1^{(j)})\right] (\tau_1^{(j)} - \tau_0^{(j)}) \in \operatorname{aff} F_{\bar\sigma_0}.$$

Now put

$$x' := x - \sum_{j=1}^{r} \left[\langle \sigma_1^{(j)}, x\rangle - \omega(\sigma_1^{(j)})\right] \tau_0^{(j)} \in \operatorname{aff} F_{\hat\sigma_1}.$$

Then $x' \equiv x \mod \tau_0$ together with

$$x' \equiv x + \sum_{j=1}^{r} \left[\langle \sigma_1^{(j)}, x\rangle - \omega(\sigma_1^{(j)})\right] (\tau_1^{(j)} - \tau_0^{(j)}) \mod \tau_1$$

imply the desired formula. □

We see that if (σ, τ) is a smooth pair then the monodromy along any primary loop $(\sigma_0\tau_0\sigma_1\tau_1)$ for minimal transversal $\sigma_0, \sigma_1 \subset \sigma$ and $\tau_0, \tau_1 \subset \tau$ is trivial. Thus, the affine structure could be first defined on the U_σ's and V_τ's and then extended to the neighborhoods of the smooth vertices. This phenomenon does not occur in the hypersurface case. We cannot however extend the affine structure across the non-smooth $(\hat\sigma, \hat\tau)$ since there are primary loops in the neighborhoods with non-trivial monodromy.

Corollary 3.2. *In a neighborhood of a vertex in D the linear part of the monodromy*

$$\operatorname{Lin}(T)\colon \pi_1(\operatorname{Star}_\Sigma(\hat\sigma_\Delta \times \hat\tau_\nabla)\backslash D) \to \mathrm{SL}_{d-r}(\mathbb{Z})$$

in a suitable basis is represented by a subgroup of the (abelian) group of matrices in the form

$$\begin{pmatrix} \mathrm{id} & * \\ 0 & \mathrm{id} \end{pmatrix}$$

(the $(d-r)\times(d-r)$*-identity matrix plus a* $(\dim(\sigma)-r+1)\times(\dim(\tau)-r+1)$ *block in the upper right corner).*

Proof. Fix a minimal tranversal cell $\sigma_0 \subset \sigma$, and consider $\mathrm{Lin}(T)$ as an endomorphism of the tangent space $\sigma_0^\perp$ of aff $F_{\hat\sigma_0}$. The fundamental group $\pi_1(\mathrm{Star}_\Sigma(\hat\sigma_\Delta \times \hat\tau_\nabla) \setminus D)$ is generated (not freely) by the primary loops $(\sigma_0\tau_p\sigma_k\tau_q)$ for minimal $\sigma_k \subset \sigma, \tau_{p,q} \subset \tau$.

Let $W_\tau \subset \sigma_0^\perp$ be the tangent space of τ_Δ, i.e., the linear span (of dimension $\dim\tau - r + 1$) of the $\{\tau_p^{(j)} - \tau_q^{(j)}\}_{j=1}^r$, all minimal $\tau_{p,q} \subset \tau$. If we extend an integral basis for W_τ to an integral basis of $\sigma_0^\perp$, then, because $W_\tau \subset \sigma_k^\perp$, the linear part

$$\mathrm{Lin}(T_{(\sigma_0\tau_p\sigma_k\tau_q)})(x) = x + \sum_{j=1}^r \langle \sigma_k^{(j)}, x\rangle (\tau_q^{(j)} - \tau_p^{(j)})$$

of the monodromy along any primary loop $(\sigma_0\tau_p\sigma_k\tau_q)$ will have the desired form. □

Corollary 3.3. *For a primary loop* $(\sigma_0\tau_0\sigma_1\tau_1)$*, the following are equivalent.*

1. *The monodromy around* $(\sigma_0\tau_0\sigma_1\tau_1)$ *is trivial.*
2. *There is a smooth pair* (σ,τ) *with* $\sigma_0, \sigma_1 \subset \sigma$ *and* $\tau_0, \tau_1 \subset \tau$.
3. *For some* j, $\sigma_0^{(j)} \neq \sigma_1^{(j)}$ *and* $\tau_0^{(j)} \neq \tau_1^{(j)}$.

Proof. We only need to show that the monodromy around non-smooth loops is non-trivial. Because σ_0 and σ_1 belong to a common face of S, we can find an $x \in \sigma_0^\perp$ so that $\langle \sigma_1, x\rangle \geq 0$, and $\langle \sigma_1^{(j)}, x\rangle > 0$. Analogously, choose $y \in \tau_0^\perp$ so that $\langle y, \tau_1\rangle \geq 0$, and $\langle y, \tau_1^{(j)}\rangle > 0$. Then

$$\langle y, \mathrm{Lin}(T_{(\sigma_0\tau_0\sigma_1\tau_1)})(x) - x\rangle > 0,$$

so that $\mathrm{Lin}(T_{(\sigma_0\tau_0\sigma_1\tau_1)})(x) \neq x$. □

Example 3.4. (cf. [Gro04] for details). Take the complete intersection in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ of two hypersurfaces of tridegrees (1,3,0) and (1,0,3). In $\mathbb{R}^5$ and $(\mathbb{R}^5)^*$ we let

$$\begin{aligned}
\Delta_1 = \mathrm{conv}\{&(0,-1,-1,0,0),(0,2,-1,0,0),(0,-1,2,0,0),\\
&(1,-1,-1,0,0),(1,2,-1,0,0),(1,-1,2,0,0)\},\\
\Delta_2 = \mathrm{conv}\{&(0,0,0,-1,-1),(0,0,0,2,-1),(0,0,0,-1,2),\\
&(-1,0,0,-1,-1),(-1,0,0,2,-1),(-1,0,0,-1,2)\},\\
\nabla_1 = \mathrm{conv}\{&(1,0,0,0,0),(0,-1,0,0,0),(0,0,-1,0,0),(0,1,1,0,0)\},\\
\nabla_2 = \mathrm{conv}\{&(-1,0,0,0,0),(0,0,0,-1,0),(0,0,0,0,-1),(0,0,0,1,1)\}.
\end{aligned}$$

For simplicity we choose $\omega = \mathbb{1}$ and $\nu = \mathbb{1}$. Then Σ is S^3 and the discriminant locus is the Hopf link of two circles each with multiplicity 12. The complement $\Sigma \setminus D$ is homotopy equivalent to a 2-torus. The global monodromy $\mathbb{Z}^2 \to \mathrm{SL}_3(\mathbb{Z})$ can be represented by abelian matrices:

$$(a, b) \mapsto \begin{pmatrix} 1 & 12a & 12b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.3. **A glimpse of mirror symmetry.** The dual affine structure arises from interchanging the rôles of Δ, ω and ∇, ν. The topological space Σ and the discriminant remain literally the same. The linear part of the monodromy is given by the transpose inverse matrices if one choose the dual bases. This constitutes the topological part of the duality.

The geometric part is much more involved. Recall that the affine structure (not just its monodromy representation) depends on the map $\phi : \Sigma \to \nabla_\omega$. We can also consider its dual counterpart - the map $\phi^\vee : \Sigma \to \Delta_\nu$. To define an integral *Kähler* affine structure we need the composition $\phi^\vee \circ \phi^{-1} : \nabla_\omega \to \Delta_\nu$ be locally potential. The affine Calabi conjecture says that for every pair of polarizations (ω, ν) there is a unique choice for $\phi^\vee \circ \phi^{-1}$ such that the local potentials solve the real Monge-Ampère equation.

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